

TEST CONFIGURATIONS, LARGE DEVIATIONS AND GEODESIC RAYS ON TORIC VARIETIES

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ABSTRACT. This article contains a detailed study in the case of a toric variety of the geodesic rays φ_t defined by Phong-Sturm corresponding to test configurations T in the sense of Donaldson. We show that the ‘Bergman approximations’ $\varphi_k(t, z)$ of Phong-Sturm converge in C^1 to the geodesic ray φ_t , and that the geodesic ray itself is $C^{1,1}$ and no better. In particular, the Kähler metrics $\omega_t = \omega_0 + i\partial\bar{\partial}\varphi_t$ associated to the geodesic ray of potentials are discontinuous across certain hypersurfaces and are degenerate on certain open sets.

A novelty in the analysis is the connection between Bergman metrics, Bergman kernels and the theory of large deviations. We construct a sequence of measures μ_k^z on the polytope of the toric variety, show that they satisfy a large deviations principle, and relate the rate function to the geodesic ray.

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Date: February 2, 2008.

Research partially supported by NSF grants DMS-0604805 and DMS-0603850.

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1. INTRODUCTION

This article is inspired by recent work of Phong-Sturm [PS2] on *test configurations* and geodesic rays for ample line bundles $L \rightarrow M$ over Kähler manifolds (M, ω) . The main construction in [PS2] associates to a test configuration $T = (\mathcal{L} \rightarrow \chi \rightarrow \mathbb{C})$ and a hermitian metric h_0 on L an infinite geodesic ray $R(h_0, T) = e^{-\psi_t} h_0$ starting at h_0 in the infinite dimensional symmetric space \mathcal{H} of hermitian metrics on L in a fixed Kähler class in the sense of Mabuchi, Semmes and Donaldson [M, S, D1]. At this time, it seems to be the only known construction of an infinite geodesic ray with given initial point in the incomplete space \mathcal{H} (see also [AT] for other constructions of geodesic rays). The geodesic rays in [PS2] are constructed by taking limits of ‘Bergman geodesic rays’, i.e. geodesic rays in the finite dimensional symmetric spaces \mathcal{B}_k of Bergman metrics. The purpose of this paper is to analyze the construction in detail in the case of toric test configurations on toric varieties. We give explicit formulae for the Bergman geodesic rays and for the limit geodesic ray. The formulae show clearly that the geodesic rays produced by toric test configurations are $C^{1,1}$ and not C^2 , and that the approximating Bergman geodesics converge in $C^1([0, L] \times M)$ for any $L > 0$. Furthermore, the metrics $\omega_t = \omega_0 + i\partial\bar{\partial}\psi_t$ are only semi-positive for $t > 0$, i.e. $\omega_t^m = 0$ on certain open sets (cf. Theorem 1.4). Hence, both in terms of regularity and positivity, the rays lie in some sense on the boundary of \mathcal{H} , and we obtain a weak solution of the Monge-Ampère equation which saturates the known $C^{1,1}$ regularity results [Ch].

Two other articles have recently appeared which contain regularity results on test configuration geodesic rays. First, $C^{1,1}$ geodesic rays are constructed by Phong-Sturm [PS3] from any test configuration using a resolution of singularities. At least when the total space of the test configuration is smooth, the rays must be the limits of Bergman rays. In our toric setting, the total space is never smooth and it is not clear at present how our regularity results overlap. Second, the $C^{1,1}$ regularity of test configuration geodesic rays with smooth total space was also observed in the article [CT] of Chen-Tang. The authors also give examples of toric test configuration geodesic rays which are not smooth [CT].

In proving the convergence result, we employ a novel connection between analysis on toric varieties and the theory of large deviations. As we will show in Theorems 1.4 and 8.1, the Phong-Sturm geodesic ray ψ_t arises from Varadhan's Lemma applied to a family $\{\mu_k^z\}$ of probability measures on the polytope P of M which are defined by the test configuration. It is closely related to the rate functional of a large deviations principle for another ‘time-tilted’

family $\mu_k^{z,t}$ of probability measures on P . We believe that this connection is of independent interest and therefore develop it in some depth for its own sake. The measures μ_k^z are closely related to the random variables and probability measures on \mathbb{R} defined in [PS] for the geodesic problem on a general Kähler variety (see the first remark in Section 5). To be precise, the Phong-Sturm measures are the pushforwards to \mathbb{R} under a certain function of the μ_k^z . In subsequent work, we study the asymptotic properties and large deviations properties of the \mathbb{R} measures on any Kähler manifold.

To state our results, we need some notation. Let M be a smooth m -dimensional toric variety, let \mathbf{T}^m denote the real torus $(S^1)^m$ which acts on M , and let $L \rightarrow M$ be a very ample toric line bundle. Let $h_0 \in \mathcal{H}$ be a positively curved reference metric on L in the give Kähler class, let ω_0 denote its curvature $(1,1)$ form and let μ_0 denote the moment map for the \mathbf{T}^m action on M with respect to μ_0 . We denote by T a test configuration. In the case of a toric variety, Donaldson [D1] shows that general toric test configurations are determined by rational piecewise linear convex functions

$$f = \max\{\lambda_1, \dots, \lambda_p\}, \quad \text{with } \lambda_j(x) = \langle \nu_j, x \rangle + v_j \quad (1)$$

on the polytope P of the toric variety, where $\lambda_j(x)$ are affine-linear functions with rational coefficients. Roughly speaking, the graph of $R - f(x)$ for a large integer $R > 0$ is the ‘top’ of an $m+1$ -dimensional polytope Q with base the m -dimensional polytope P and the degeneration occurs as one moves from the bottom to the top. By multiplying f by d we may assume the affine functions λ_j have integral coefficients. We denote by $P_j \subset P$ the subdomain where $f = \lambda_j$.

In [PS2], the geodesic ray $e^{-\psi_t} h_0$ is constructed as a limit of Bergman geodesic rays $h(t; k) = h_0 e^{-\psi_k(t, z)}$ which are constructed from the test configuration T (see Definition 3). In the following Proposition, we given an exact formula for ψ_k in the case of a toric test configuration. It is stated in terms of monomial sections corresponding to lattice points in the polytope of M . We refer to §2 for as yet undefined terminology.

PROPOSITION 1.1. *Let (M, L, h_0, ω_0) be a polarized toric Kähler variety of dimension m , and let P denote the corresponding lattice polytope. Then the Phong-Sturm sequence of approximating Bergman geodesics is given by*

$$\psi_k(t, z) = \frac{1}{kd} \log \hat{Z}_k(t, z) \quad (2)$$

with

$$\begin{cases} \hat{Z}_k(t, z) = e^{-2t \frac{1}{d_k} \sum_{\alpha \in kdP \cap \mathbb{Z}^m} kd(R - f(\frac{\alpha}{kd}))} Z_k^{z,t}, \\ Z_k^{z,t} := \sum_{\alpha \in kdP \cap \mathbb{Z}^m} e^{2tkd(R - f(\frac{\alpha}{kd}))} \frac{\|s_\alpha(z)\|_{h_0^{kd}}^2}{\mathcal{Q}_{h_0^{kd}}(\alpha)}, \end{cases} \quad (3)$$

where $\{s_\alpha\}$ is the basis of $H^0(M, L^{kd})$ corresponding to the monomials z^α on \mathbb{C}^m with $\alpha \in kdP$, $d_k = H^0(X, L^{kd})$, and where $\mathcal{Q}_{h_0^{kd}}(\alpha)$ is the square of its L^2 norm with respect to the inner product $\text{Hilb}_{kd}(h_0)$ induced by h_0 (see (10), (11) and (23) for the precise formula).

The Phong-Sturm geodesic ray is by definition the limit $\psi_t(z)$ (in a certain topology) of the sequence $\psi_k(t, z)$. Our next result gives an explicit formula for it. One of the main points

of this article is that this relative Kähler potential is naturally expressed in terms of the *rate functions* I^z for the large deviations principle of the sequence of probability measures,

$$\mu_k^z = \frac{1}{\Pi_{h_0^{kd}}(z, z)} \sum_{\alpha \in kdP \cap \mathbb{Z}^m} \frac{|s_\alpha(z)|_{h_0^{kd}}^2}{\|s_\alpha\|_{h_0^{kd}}^2} \delta_{\frac{\alpha}{kd}}, \quad (4)$$

where $\Pi_{h_0^{kd}}(z, z)$ is the contracted Szegö kernel on the diagonal (or density of states); see §2 for background. We obviously have

$$\psi_k(t, z) = \frac{1}{kd} \log \int_P e^{kdt(R-f(x))} d\mu_k^z(x) + 2t \frac{1}{d_k} \sum_{\alpha \in kdP \cap \mathbb{Z}^m} kd(R - f(\frac{\alpha}{kd})), \quad (5)$$

and since the second term has an obvious limit, the determination of ψ_t reduces to the uniform asymptotics of the first term. For notational simplicity we henceforth often write

$$F_t(x) = t(R - f(x)). \quad (6)$$

Our first observation is the following measure concentration result, which follows easily from the Bernstein polynomial results of [Ze2].

PROPOSITION 1.2. *Let $\mu_0 : M \rightarrow P$ be the moment map with respect to the symplectic form ω_0 . Then for any $z \in M$, the measures μ_k^z tend weakly to $\delta_{\mu_0(z)}$. Thus,*

$$\mu_0(z) = \lim_{k \rightarrow \infty} \frac{1}{\Pi_{h_0^{kd}}(z, z)} \sum_{\alpha \in kdP} \left(\frac{\alpha}{kd}\right) \frac{\|s_\alpha(z)\|_{h_0^{kd}}^2}{\mathcal{Q}_{h_0^{kd}}(\alpha)}.$$

A much deeper result is that μ_k^z satisfy a large deviations principle (LDP), and that the logarithmic asymptotics in (5) are therefore determined by Varadhan's Lemma. Heuristically, an LDP means that the measure $\mu_k^z(A)$ of a Borel set A is obtained asymptotically by integrating $e^{-kI^z(x)}$ over A , where I^z is known as the rate functional and k is the rate. The rate functions I^z for $\{d\mu_k^z\}$ depend on whether z lies in the open orbit M^o of M or on the divisor at infinity \mathcal{D} ; equivalently, they depend on whether the image $\mu_0(z)$ of z under the moment map for ω_0 lies in the interior P^o of the polytope P or along a face F of its boundary ∂P . The definition of uniformity of the Laplace large deviations principle will be given in section §6. The notation and terminology will be defined and reviewed in §2.

THEOREM 1.3. *For any $z \in M$, the probability measures μ_k^z satisfy a uniform Laplace large deviations principle with rate k and with convex rate functions $I^z \geq 0$ on P defined as follows:*

- If $z \in M^o$, the open orbit, then $I^z(x) = u_0(x) - \langle x, \log |z| \rangle + \varphi_{P^o}(z)$, where φ_{P^o} is the canonical Kähler potential of the open orbit and u_0 is its Legendre transform, the symplectic potential;
- When $z \in \mu_0^{-1}(F)$ for some face F of ∂P , then $I^z(x)$ restricted to $x \in F$ is given by $I^z(x) = u_F(x) - \langle x', \log |z'| \rangle + \varphi_F(z)$, where $\log |z'|$ are orbit coordinates along F , φ_F is the canonical Kähler potential for the subtoric variety defined by F and u_F is its Legendre transform. On the complement of \bar{F} it is defined to be $+\infty$.
- When z is a fixed point then $I^z(v) = 0$ and elsewhere $I^z(x) = \infty$.

The large deviations principle seems to us of independent interest. The statement for fixed z follows from the Gärtner-Ellis theorem (cf. [DZ, dH, E]), once it is established that the hypotheses of this theorem are satisfied. In addition, we prove that the upper and lower bounds are in a certain sense uniform in the z parameter. This is a rather complicated matter in our setting, since the rate functions I^z are highly non-uniform, and we follow the definition of Laplace large deviations principles of [DE] in defining uniform large deviations. They are precisely adapted to Varadhan's Lemma and will imply that $\varphi_k \rightarrow \varphi$ in C^0 . From this large deviations result and a more substantial one below (Theorem 8.1), we obtain our convergence result:

THEOREM 1.4. *Let $L \rightarrow X$ be a very ample toric line bundle over a toric Kähler manifold. Let h_0 be a positively curved metric on L and T a test configuration. Then $\psi_k(t; z)$ converges in C^1 to the $C^{1,1}$ geodesic ray $\psi_t(z)$ in \mathcal{H} given by (cf. (6))*

$$\psi_t(z) = \sup_{x \in P} [F_t - I^z] \quad (7)$$

Moreover, ψ_t has a bounded but discontinuous second spatial and t derivatives and $\omega_t = \omega_0 + \partial\bar{\partial}\psi_t$ has a zero eigenvalue in certain open sets. Explicitly

- When $z \in M^\circ$, the open orbit, $\psi_t(z) = \mathcal{L}_{\mathbb{R}^m}(u_0 + tf) - \varphi_{P^\circ}$ where $\mathcal{L}_{\mathbb{R}^m}$ is the Legendre transform on \mathbb{R}^m . There is a simpler formula which uses the moment map μ_t associated to ω_t which is introduced in Proposition 1.5. As will be shown in Proposition 7.15, in the region $\mu_t^{-1}(P_j)$, we have $\psi_t(e^{\rho/2+i\theta}) = \varphi_{P^\circ}(e^{(\rho+tv_j)/2+i\theta}) - tv_j - \varphi_{P^\circ}$, where $z = e^{\rho/2+i\theta}$.
- When $z \in \mu_0^{-1}(F^\circ)$, then $\psi_t(z) = \mathcal{L}_{F^\circ}(u_F + tf) - \varphi_F$ where \mathcal{L}_{F° denote the Legendre transform on the quotient of \mathbb{R}^m by the isotropy subgroup of z ;
- When $z = \mu_0^{-1}(v)$ is a vertex, then $\psi_t(z) = -tf(v)$.
- A point $z = e^{\rho/2+i\theta} \in \mu_0^{-1}(P_j \cap P_k)$ only if $\rho \in tCH(\nu_j, \nu_k)$, where CH denotes the convex hull. In that case, $\mu_t(\rho)$ is a constant point x_0 for $\rho \in tCH(\nu_j, \nu_k)$, and $\psi_t(z) = \langle \rho, x_0 \rangle - u_0(x_0) - t(\langle \nu_j, x_0 \rangle + v_j) - \varphi_{P^\circ}(z)$. In these open sets, $\omega_t^m \equiv 0$. Analogous formulae hold on the faces of ∂P .

We obtain the formula for ψ_t by applying Varadhan's Lemma to the integrals (5). Uniformity of the limit is a novel feature. An additional part of the proof of Theorem 1.4 is to show that (7) is in fact a $C^{1,1}$ function on M . Even on the open orbit, it requires some convex analysis to see that $\mathcal{L}(u_0 + tf) \in C^{1,1}(\mathbb{R}^m)$. Roughly speaking, the Legendre transforms smooths out the corners of f to C^1 , but no further than $C^{1,1}$. We must then verify that the extension to M of ψ_t remains $C^{1,1}$. Since ψ_t is C^1 it determines a moment map, given over the open orbit by

$$\mu_t : M^\circ \rightarrow P, \quad \mu_t(e^{\rho/2+i\theta}) = \nabla_\rho \psi_t(e^{\rho/2+i\theta}) \text{ on } M^\circ. \quad (8)$$

An interesting feature of the convex analysis is that μ_t fails to be a homeomorphism from M/\mathbf{T}^m to P as in the smooth case. Indeed, the usual inverse map defined by gradient of the symplectic potential pulls apart the polytope discontinuously into different regions. However, in §7.2 we explain how the moment map of the singular metrics ψ_t is rather a homeomorphism from the underlying real toric variety $M_{\mathbb{R}}$ (cf. §2) to the graph of the subdifferential of $u + tf$. More precisely, this statement is correct on the open orbit and has a natural closure on the boundary.

The proof of C^0 convergence to the limit is already an improvement on the degree of convergence in [PS2]. To prove C^1 convergence, we consider integrals of bounded continuous functions on P against the ‘time-tilted’ probability measures

$$d\mu_k^{z,t} := \frac{1}{Z_k^{t,z}} \sum_{\alpha \in kdP \cap \mathbb{Z}^m} e^{2kd(F_t(\frac{\alpha}{kd}))} \frac{\|s_\alpha(z)\|_{h_0^{kd}}^2}{Q_{h_0^{kd}}(\alpha)} \delta_{\frac{\alpha}{k}}. \quad (9)$$

It follows from the ‘tilted’ Varadhan Lemma and from Theorem 1.3 that the sequence $d\mu_k^{z,t}$ of probability measures on P satisfies a large deviations principle with convex rate function

$$I^{z,t} = I^z + tf(x) + \sup_{x \in P} [F_t - I^z].$$

On the open orbit, $I^{z,t}(x) = -\langle x, \log |z| \rangle + u_0(x) + \varphi_{P^o}(\rho)$. Thus, the Kähler potential of the singular metric corresponding to $u + tf$, given in Theorem 1.4 (see (7)), is closely related to the rate functional for $\mu_k^{z,t}$.

The importance of these tilted measures is that the first derivatives $d\psi_k(t, z)$ in t and z variables can be expressed as integrals of continuous functions on the compact set P against $d\mu_k^{t,z}$. We prove the following parallel to Proposition 1.2:

PROPOSITION 1.5. *For all (z, t) , there exists a unique limit point $\mu_t(z) \in P$*

$$\mu_t(z) = \lim_{k \rightarrow \infty} \int_P x d\mu_k^{z,t},$$

which is the same as (8). We have $\mu_k^{z,t} \rightarrow \delta_{\mu_t(z)}$ in the weak sense as $k \rightarrow \infty$. Further, $d_{z,t}\psi_k(t, z) \rightarrow d_{z,t}\psi(t, z)$ uniformly as $k \rightarrow \infty$. More precisely,

$$\begin{cases} d\psi_k(t, z) \rightarrow \bar{x}(t, z) = \lim_{k \rightarrow \infty} \int_P x d\mu_k^{z,t}, \\ \dot{\psi}_k(t, z) \rightarrow R - f(\bar{x}(t, z)) = \lim_{k \rightarrow \infty} \int_P (R - f(x)) d\mu_k^{z,t}. \end{cases}$$

The organization of this article is as follows: After reviewing the basic geometric and analytic objects in §2, we introduce the measures $d\mu_k^z$ in §4 and prove Proposition 1.2 and state Theorem 1.3. In §5 we begin the proof of Theorems 1.3 by studying the scaling limit of the logarithmic moment generating function of $d\mu_k^z$. In §5.2 we prove Theorem 1.3. In §3, we introduce toric test configurations and prove Proposition 1.1. In §6, we prove the C^0 convergence part of Theorem 1.4 using Varadhan’s Lemma (Theorem 6.1). This proves the explicit formula (7). We then study the regularity of ψ_t in §7.2, define the moment map μ_t and study its regularity and mapping properties. The analysis shows that $\psi_t \in C^1(M)$ for each t . In §8 we show that $\psi_k \rightarrow \psi$ in $C^1([0, L] \times M)$ for any $L > 0$. We cannot of course obtain convergence in C^2 since $\psi_t \notin C^2$.

In conclusion, we would like to thank D.H. Phong and J. Sturm for many discussions of the subject of this article as it evolved. We also thank V. Alexeev for tutorials on toric test configurations and in particular for corroborating Proposition 3.2. We thank O. Zeitouni for tutorials in the theory of large deviations. The initial results of this paper were presented at the Complex Analysis and Geometry XVIII conference in Levico in May, 2007.

2. BACKGROUND ON TORIC VARIETIES

We employ the same notation and terminology as in [D1, SoZ1, SoZ2]. We briefly recall the main definitions for the reader's convenience.

We recall that a toric Kähler manifold is a Kähler manifold (M, J, ω_0) on which the complex torus $(\mathbb{C}^*)^m$ acts holomorphically with an open orbit M^o . We choose a basepoint m_0 on the orbit open and identify $M^o \equiv (\mathbb{C}^*)^m$. The underlying real torus is denoted \mathbf{T}^m so that $(\mathbb{C}^*)^m = \mathbf{T}^m \times \mathbb{R}_+^m$, which we write in coordinates as $z = e^{\rho/2+i\theta}$ in a multi-index notation.

We fix the standard basis $\{\frac{\partial}{\partial \theta_j}\}$ of $\text{Lie}(\mathbf{T}^m)$ (the Lie algebra) and use the same notation for the induced vector fields on M . We also denote by $\{\frac{\partial}{\partial \rho_j}\}$ the standard basis of $\text{Lie}(\mathbb{R}_+^m)$. We then have $\frac{\partial}{\partial \rho_j} = J_0 \frac{\partial}{\partial \theta_j}$ where J_0 is the standard complex structure on \mathbb{C}^m . We use the same notation for the induced vector fields on M .

We assume that M is projective and that P is a Delzant polytope; we view M as defined by a monomial embedding. The polytope P is defined by a set of linear inequalities

$$l_r(x) := \langle x, v_r \rangle - \alpha_r \geq 0, \quad r = 1, \dots, d,$$

where v_r is a primitive element of the lattice and inward-pointing normal to the r -th $(n-1)$ -dimensional face of P . We denote by P^o the interior of P and by ∂P its boundary; $P = P^o \cup \partial P$. We normalize P so that $0 \in P$ and $P \subset \mathbb{R}_{\geq}^m$. Here, and henceforth, we put $\mathbb{R}_{\geq} = \mathbb{R}_+ \cup \{0\}$. For background, see [G, A, D1, Fu].

Underlying the complex toric variety M is a real toric variety $M_{\mathbb{R}}$, namely the closure of \mathbb{R}_+^m under the monomial embedding. It is a manifold with corners homeomorphic to P (cf. [Fu], Ch. 4). Every metric moment map we consider below defines such a homeomorphism.

2.1. Monomial basis of $H^0(M, L^k)$, norms and Szegö kernels. Let $\#P$ denote the number of lattice points $\alpha \in \mathbb{N}^m \cap P$. We denote by $L \rightarrow M$ the invariant line bundle obtained by pulling back $\mathcal{O}(1) \rightarrow \mathbb{CP}^{\#P-1}$ under the monomial embedding defining M . A natural basis of the space of holomorphic sections $H^0(M, L^k)$ associated to the k th power of $L \rightarrow M$ is defined by the monomials z^α where α is a lattice point in the k th dilate of the polytope, $\alpha \in kP \cap \mathbb{N}^m$. That is, there exists an invariant frame e over the open orbit so that $s_\alpha(z) = z^\alpha e$. We denote the dimension of $H^0(M, L^k)$ by N_k . We equip L with a toric Hermitian metric $h = h_0$ whose curvature $(1, 1)$ form $\omega_0 = i\partial\bar{\partial} \log \|e\|_{h_0}^2$ lies in \mathcal{H} . We often express the norm in terms of a local Kähler potential, $\|e\|_{h_0}^2 = e^{-\psi}$, so that $|s_\alpha(z)|_{h_0}^2 = |z^\alpha|^2 e^{-k\psi(z)}$ for $s_\alpha \in H^0(M, L^k)$.

Any hermitian metric h on L induces inner products $\text{Hilb}_k(h)$ on $H^0(M, L^k)$, defined by

$$\langle s_1, s_2 \rangle_{h^k} = \int_M (s_1(z), s_2(z))_{h^k} \frac{\omega_0^m}{m!}. \quad (10)$$

The monomials are orthogonal with respect to any such toric inner product and have the norm-squares

$$Q_{h^k}(\alpha) = \int_{\mathbb{C}^m} |z^\alpha|^2 e^{-k\psi(z)} dV_\varphi(z), \quad (11)$$

where $dV_\varphi = (i\partial\bar{\partial}\varphi)^m/m!$.

The Szegö (or Bergman) kernels of a positive Hermitian line bundle $(L, h) \rightarrow (M, \omega)$ are the kernels of the orthogonal projections $\Pi_{h^k} : L^2(M, L^k) \rightarrow H^0(M, L^k)$ onto the spaces of

holomorphic sections with respect to the inner product $Hilb_k(h)$,

$$\Pi_{h^k} s(z) = \int_M \Pi_{h^k}(z, w) \cdot s(w) \frac{\omega_h^m}{m!}, \quad (12)$$

where the \cdot denotes the h -hermitian inner product at w . In terms of a local frame e for $L \rightarrow M$ over an open set $U \subset M$, we may write sections as $s = fe$. If $\{s_j^k = f_j e_L^{\otimes k} : j = 1, \dots, d_k\}$ is an orthonormal basis for $H^0(M, L^k)$, then the Szegö kernel can be written in the form

$$\Pi_{h^k}(z, w) := F_{h^k}(z, w) e^{\otimes k}(z) \otimes \overline{e^{\otimes k}(w)}, \quad (13)$$

where

$$F_{h^k}(z, w) = \sum_{j=1}^{N_k} f_j(z) \overline{f_j(w)}, \quad N_k = H^0(M, L^k). \quad (14)$$

In the case of a toric variety with $0 \in \bar{P}$, there exists a frame e such that $s_\alpha(z) = z^\alpha e$ on the open orbit, and then

$$F_{h^k}(z, w) = \sum_{j=1}^{N_k} \frac{z^\alpha \bar{w}^\alpha}{Q_{h^k}(\alpha)}. \quad (15)$$

Along the diagonal, $(F_{h^k}(z, z))^{-1}$ is a Hermitian metric. The product $F_{h^k}(z, z) \|e\|_{h^k}^2$ is then the ratio of two Hermitian metrics and it balances out to have a power law expansion,

$$\Pi_{h^k}(z, z) = \sum_{i=0}^{N_k} \|s_i^k(z)\|_{h^k}^2 = a_0 k^m + a_1(z) k^{m-1} + a_2(z) k^{m-2} + \dots \quad (16)$$

where a_0 is constant; see [T, Ze]. We note that by a slight abuse of notation, $\Pi_{h^k}(z, z)$ denotes the metric contraction of (13). It is sometimes written $B_{h^k}(z)$ and referred as the density of states. If we sift out the α th term of Π_{h^k} by means of Fourier analysis on \mathbf{T}^m , we obtain

$$\mathcal{P}_{h^k}(\alpha, z) := \frac{|z^\alpha|^2 e^{-k\psi(z)}}{Q_{h^k}(\alpha)}, \quad (17)$$

which play an important role in this article (as in [SoZ2]).

2.2. Kähler potential on the open orbit and symplectic potential. On any simply connected open set, a Kähler metric may be locally expressed as $\omega = i\partial\bar{\partial}\varphi$ where φ is a locally defined function which is unique up to the addition $\varphi \rightarrow \varphi + f(z) + \bar{f}(\bar{z})$ of the real part of a holomorphic or anti-holomorphic function f . Of course, the potential is not globally defined. We now introduce special local Kähler potentials adapted to the open orbit, respectively the divisor at infinity on a toric variety.

Without loss of generality, we assume that L is very ample. Then on the open orbit $M^o \subset M$, there is a canonical choice of the open-orbit Kähler potential once one fixes the image P of the moment map:

$$\varphi_{P^o}(z) := \log \sum_{\alpha \in P} |z^\alpha|^2 = \log \sum_{\alpha \in P} e^{\langle \alpha, \rho \rangle}. \quad (18)$$

This is the potential appearing in Theorem 1.3 for the open orbit. For instance, the Fubini-Study Kähler potential is $\varphi(z) = \log(1 + |z|^2) = \log(1 + e^\rho)$. We observe that, since $0 \in P$, (18) defines a smooth function to the full affine chart $z \in \mathbb{C}^m$ in the closure of the open orbit

chart when we use the first expression with $z = e^{\rho/2}$. As will be discussed in §2.3, this affine chart corresponds to the choice of the vertex 0 and the associated fixed point $\mu_0^{-1}(0)$. There is a corresponding affine chart and Kähler potential on the chart for any vertex.

Since it is invariant under the real torus \mathbf{T}^m -action, φ_{P^o} only depends on the ρ -variables and we have

$$\omega_0 = i \sum_{j,k} \frac{\partial^2 \varphi_{P^o}}{\partial \rho_k \partial \rho_j} \frac{dz_j}{z_j} \wedge \frac{d\bar{z}_k}{\bar{z}_k}.$$

Since ω_0 is a positive form, φ_{P^o} is a strictly convex function of $\rho \in \mathbb{R}^n$. We may view $\varphi_{P^o}(\rho)$ as a function on the Lie algebra $\text{Lie}(\mathbb{R}_+^m)$ of $\mathbb{R}_+^m \subset (\mathbb{C}^*)^m$ or equivalently as a function on the open orbit of the real toric variety $M_{\mathbb{R}}$.

The action of the real torus \mathbf{T}^m on (M, ω_0) is Hamiltonian with moment map $\mu_0 : M \rightarrow P$ with respect to ω_0 . We recall that the moment map $\mu_0 : M \rightarrow (\text{Lie}(\mathbf{T}^m))^*$ is defined by $\langle \mu(z), X \rangle = H_X(z)$ where H_X is the Hamiltonian of X^* ; X^* is the induced Hamiltonian vector field on M induced by natural map $X \in \text{Lie}(\mathbf{T}^m)$. Over the open orbit, the moment map may be expressed as

$$\mu_0(z_1, \dots, z_m) = \left(\frac{\partial \varphi_{P^o}}{\partial \rho_1}, \dots, \frac{\partial \varphi_{P^o}}{\partial \rho_m} \right); \quad (z = e^{\rho/2+i\theta}). \quad (19)$$

Although the right side is an expression in terms of the locally defined Kähler potential φ_{P^o} , which is singular ‘at infinity’, the components $\frac{\partial \varphi_{P^o}}{\partial \rho_j}$ extend to all of M as smooth functions. This follows from the fact that μ_0 is globally smooth. For instance, for the Fubini-Study metric on \mathbb{CP}^m , the moment map is $\mu_0(z) = \frac{(|z_1|^2, |z_2|^2, \dots, |z_m|^2)}{1 + \|z\|^2}$, where $\|z\|^2 = |z_1|^2 + \dots + |z_m|^2$ and the Kähler potential on the open orbit is $\varphi_{P^o}(\rho) = \log(1 + e^{\rho_1} + \dots + e^{\rho_m})$, with $\frac{\partial \varphi_{P^o}}{\partial \rho_j} = \frac{e^{\rho_j}}{(1 + e^{\rho_1} + \dots + e^{\rho_m})}$.

The moment map defines a homeomorphism $\mu_0 : M_{\mathbb{R}} \rightarrow P$. Later we will need to define the inverse of the map (19) on $M_{\mathbb{R}}$ so we take some care at this point to make explicit the identifications implicit in the formula. First, we decompose $\text{Lie}(\mathbb{C}^*)^m = \text{Lie}(\mathbf{T}^m) \oplus \text{Lie}(\mathbb{R}_+^m)$. Viewing φ_{P^o} as a function on $\text{Lie}(\mathbb{R}_+^m)$, $d\varphi_{P^o}(\rho) \in T_{\rho}^* \text{Lie}(\mathbb{R}_+^m) \simeq \text{Lie}(\mathbb{R}_+^m)^*$. Under $J_0 : \text{Lie}(\mathbb{R}_+^m)^* \simeq \text{Lie}(\mathbf{T}^m)^*$ so we may regard $d\varphi_{P^o} : M_{\mathbb{R}} \rightarrow \text{Lie}(\mathbf{T}^m)^*$ as the moment map.

We now consider the *symplectic potential* u_0 associated to φ_{P^o} , defined as the Legendre transform of φ_{P^o} on \mathbb{R}^m :

$$u_0(x) = \varphi_{P^o}^*(x) = \mathcal{L}\varphi_{P^o}(x) := \sup_{\rho \in \mathbb{R}^m} (\langle x, \rho \rangle - \varphi_{P^o}(e^{\rho/2+i\theta})). \quad (20)$$

It is a function on P , or in invariant terms it is a function on $\text{Lie}(\mathbf{T}^m)^* \simeq \text{Lie}(\mathbb{R}_+^m)^*$. In general, the Legendre transform of a function on a vector space V is a function on the dual space V^* . The symplectic potential has canonical logarithmic singularities on ∂P . According to [A] (Proposition 2.8) or [D1] (Proposition 3.1.7),

$$u_0(x) = \sum_k \ell_k(x) \log \ell_k(x) + f_0 \quad (21)$$

where $f_0 \in C^\infty(\bar{P})$.

The differential du_0 in a sense defines a partial inverse for the moment map. More precisely, we have

$$\left(\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_m}\right) = 2 \log \mu_0^{-1}(x) \iff \mu_0^{-1}(x) = \exp \frac{1}{2} \left(\frac{\partial u_0}{\partial x_1}, \dots, \frac{\partial u_0}{\partial x_m}\right), \quad (22)$$

where $\exp : \text{Lie}(\mathbb{R}_+^m) \rightarrow \mathbb{R}_+^m$ is the exponential map. In coordinates, this follows from the fact that

$$u_0(x) = \langle x, \rho \rangle - \varphi_{P^o}(e^{\rho/2+i\theta}) \implies \nabla u_0(x) = \rho - \langle x, \nabla_x \rho \rangle - \langle \nabla \varphi_{P^o}(e^{\rho/2+i\theta}), \nabla_x \rho \rangle = \rho,$$

when $\nabla \varphi_{P^o}(e^{\rho/2+i\theta}) = x$. To interpret (22) invariantly, we note that $du_0(x) \in T_x^*(\text{Lie}(\mathbf{T}^m)^*) \simeq \text{Lie}(\mathbf{T}^m) \simeq \text{Lie}(\mathbb{R}_+^m)$ while $\mu_0^{-1}(x) \in M_{\mathbb{R}}^o \simeq \mathbb{R}_+^m$ so that $2 \log \mu_0^{-1}(x) \in \text{Lie}(\mathbb{R}_+^m)$. We observe that (22) defines an inverse of μ_0 from the open orbit of the base point under R_+^m to its image P^o and that it extends to a homeomorphism between the manifolds with corners $\mathbb{R}_{\geq}^m \iff \mu_0(\mathbb{R}_{\geq}^m)$. Here, $\mathbb{R}_{\geq} = R_+ \cup \{0\}$.

It will also be important to write the norming constants in terms of the symplectic potential:

$$\mathcal{Q}_{h^k}(\alpha) = \int_P e^{k(u_0(x) + \langle \frac{\alpha}{k} - x, \nabla u_0(x) \rangle)} dx. \quad (23)$$

It follows from [SoZ2] (Proposition 3.1) and from [STZ] that for interior α , and α_k with $|\alpha - \alpha_k| = O(\frac{1}{k})$,

$$\mathcal{Q}_{h^k}(\alpha_k) \sim k^{-m/2} e^{ku_0(\alpha)}, \quad (24)$$

and for all α and α_k with $|\alpha - \alpha_k| = O(\frac{1}{k})$ that

$$\frac{1}{k} \log \mathcal{Q}_{h^k}(\alpha_k) = u_0(\alpha) + O\left(\frac{\log k}{k}\right). \quad (25)$$

2.3. The divisor at infinity \mathcal{D} and the boundary ∂P of P . The above definitions concern the behavior of the Kähler potential on the open orbit $(\mathbb{C}^*)^m$ and the dual behavior of the symplectic potential. As noted above, the Kähler potential extends smoothly to the full affine chart \mathbb{C}^m . This is but one affine chart needed to cover M in the distinguished atlas $\{U_v\}$ parameterized by vertices v of P . We briefly explain how to modify the above constructions so that they apply to the other charts, referring to [SoZ1, STZ] for further discussion.

For each vertex $v \in P$, we define the chart U_v by $U_v := \{z \in M ; s_v(z) \neq 0\}$, where s_v is the monomial section corresponding to v . Since P is Delzant, there exist $\alpha^1, \dots, \alpha^m \subset \mathbb{N}^m \cap P$ such that each α^j lies on an edge incident to v , and the vectors $v^j := \alpha^j - v$ form a basis of \mathbb{Z}^m . We define

$$\eta_v : (\mathbb{C}^*)^m \rightarrow (\mathbb{C}^*)^m, \quad \eta_v(z) := (z^{v^1}, \dots, z^{v^m}). \quad (26)$$

The map η is a \mathbf{T}^m -equivariant biholomorphism of $(\mathbb{C}^*)^m$ with inverse

$$z : (\mathbb{C}^*)^m \rightarrow (\mathbb{C}^*)^m, \quad z(\eta) = (\eta^{\Gamma e^1}, \dots, \eta^{\Gamma e^m}), \quad (27)$$

where e^j is the standard basis for \mathbb{C}^m , and Γ is defined by

$$\Gamma v^j = e^j, \quad v^j = \alpha^j - v. \quad (28)$$

The corner of P at v is transformed to the standard corner of the orthant \mathbb{R}_+^m by the affine linear transformation

$$\tilde{\Gamma} : \mathbb{R}^m \ni u \rightarrow \Gamma u - \Gamma v \in \mathbb{R}^m, \quad (29)$$

which preserves \mathbb{Z}^m , carries P to a polytope $Q_v \subset \{x \in \mathbb{R}^m ; x_j \geq 0\}$ and carries the facets F_j incident at v to the coordinate hyperplanes $= \{x \in Q_{v_0} ; x_j = 0\}$. The map η extends a homeomorphism: $\eta : U_v \rightarrow \mathbb{C}^m$, and

$$\eta(\mu_P^{-1}(\bar{F}_j)) = \{\eta \in \mathbb{C}^m ; \eta_j = 0\}.$$

For each v we then define the Kähler potential φ_{U_v} on $U_v \simeq \mathbb{C}^m$ by

$$\varphi_{U_v}(\eta) = \log \sum_{\alpha \in P} |\eta^{\tilde{\Gamma}(\alpha)}|^2. \quad (30)$$

The Legendre transform of φ_{U_v} as a function on \mathbb{R}^m defines a dual symplectic potential u_{U_v} on P . Generalizing (22), the inverse of the moment map may be expressed near the corner at v by

$$\mu_0^{-1}(x) = \exp \frac{1}{2} \left(\frac{\partial u_{U_v}}{\partial x_1}, \dots, \frac{\partial u_{U_v}}{\partial x_m} \right), \quad (31)$$

where the right side is identified with a point in $U_v \cap M_{\mathbb{R}}$. Thus (31) defines a homeomorphism from the corner at v to its inverse image under μ_0 .

To illustrate the notation in the simplest example of \mathbb{CP}^1 with its Fubini-Study metric and with $v = 1$ we note that $\eta_v(z) = z^{-1}$, $\tilde{\Gamma}(u) = 1 - u$, $\varphi_{U_1}(\eta) = \log(1 + |\eta|^2)$, and $u_{U_1}(y) = y \log y + (1 - y) \log(1 - y)$. On the overlap $U_0 \cap U_1$, we have $du_{U_0}(x) = -du_{U_1}(y)$. Indeed, $y = \tilde{\Gamma}(x)$ and so $du(y) = \log \frac{y}{1-y} = \log \frac{1-x}{x} = -du(x)$. Hence, $e^{\frac{1}{2}du(y)} = \frac{y}{1-y}$ is the inverse of $e^{\frac{1}{2}du(x)}$ and μ_0^{-1} is locally expressed as a map from a neighborhood of $v = 1$ up to $y = 0$.

We also need to discuss moment maps and Kähler potentials for the toric sub-varieties corresponding to boundary faces. As in [Fu, SoZ1, STZ], a face of P is the intersection of P with a supporting affine hyperplane; a top $m - 1$ -dimensional face is a facet; while at the other extreme, the lowest dimensional faces are the vertices. We denote the relative interior of a face by F° . Each face defines a sub-toric variety $M_F = \mu^{-1}(F) \subset \mathcal{D}$. This subtoric variety also has an open orbit and a moment map. In particular, over the open orbit $\mu^{-1}(F^\circ)$, there is a canonical Kähler potential for $\omega_0|_{M_F}$:

$$\varphi_F(z) = \log \sum_{\alpha \in \overline{F}} |z^\alpha|^2 = \log \sum_{\alpha \in \overline{F}} e^{\langle \alpha, \rho'' \rangle}, \quad (32)$$

where now $\rho'' \in \mathbb{R}^{m-k}$ if $\dim \mathbf{T}_z^m = k$. Further the Legendre transform of φ_F on \mathbb{R}^{m-k} defines a symplectic potential $u_F(x'')$ along F . Note that $u_0 = 0$ on ∂P , so u_F is not the restriction of u_0 to F . These φ_F, u_F appear in Theorem 1.3 in the formula for the rate function when $z \in F$. In the extreme case of a vertex v corresponding to a fixed point of the $(\mathbb{C}^*)^m$, $\varphi_v = 0$.

2.4. Summary of Kähler potentials. We summarize the different notions of Kähler potential we have introduced:

- ψ_t is the relative Kähler potential with respect to h_0 for the geodesic ray $h_t = e^{-\psi_t} h_0$. It is globally defined on M and $\psi_0 = 0$;
- The Bergman geodesic ray potentials $\psi_k(t, z)$ (see Definition 3) are also relative Kähler potentials, with respect to h_k arising from $Hilb_k(h_0)$. They are also globally defined on M and are $o(1)$ as $k \rightarrow \infty$ at $t = 0$;
- $\varphi_{P^\circ}(z)$ is the open orbit Kähler potential corresponding to h_0 , i.e. it is the potential for ω_0 on the open orbit. Similarly, φ_F is the potential valid near $\mu_0^{-1}(F)$.

3. TORIC TEST CONFIGURATIONS

The purpose of this section is to give the proof of Proposition 1.1. We include the basic definitions on toric test configurations for the sake of completeness. Proposition 1.1 is then simple to prove and is known to experts; the statement can also be found in [ZZ].

We first recall that a test configuration as defined by Donaldson [D1] consists of the following:

- A scheme χ with a \mathbb{C}^* action ρ ;
- A \mathbb{C}^* equivariant line bundle $\mathcal{L} \rightarrow \chi$ which is ample on all fibers;
- A flat \mathbb{C}^* equivariant map $\pi : \chi \rightarrow \mathbb{C}$ where \mathbb{C}^* acts on \mathbb{C} by multiplication.
- The fiber X_1 is isomorphic to X and (X, L^r) is isomorphic to (X_1, L_1) where for $w \in \mathbb{C}$, $X_w = \pi^{-1}(w)$ and $L_w = \mathcal{L}|_{X_w}$.

For this article, only the *weights* η_α of the test configuration play a role, i.e. the weights of the \mathbb{C}^* action on $H^0(X_0, L_0^k)$ where X_0 is the central fiber (i.e. the eigenvalues of B_k in the notation of [PS2]). We define the normalized weights (i.e. the eigenvalues of the traceless part A_k of B_k in the notation of [PS2]) by

$$\lambda_\alpha = \eta_\alpha - \frac{1}{N_k} \sum_{j=1}^{N_k} \eta_j \quad (33)$$

The geodesic ray associated to the test configuration is defined in terms of the normalized weights as follows:

Definition: The Phong-Sturm test configuration geodesic ray is the weak limit of the Bergman geodesic rays $h(t; k) : (-\infty, 0) \rightarrow \mathcal{H}_k$ given by

$$h(t; k) = h_{\hat{s}(t, k)} = h_0 e^{-\psi_k(t)},$$

with

$$\psi_k(t, z) = \frac{1}{k} \log \left(k^{-n} \sum_{\alpha=0}^{N_k} e^{2t\lambda_\alpha} \frac{|s_\alpha(z)|_{h_0^k}^2}{\|s_\alpha\|_{h_0^k}^2} \right)$$

3.1. Calculation of the weights. In this section, we outline the calculation of the weights in the case of a toric test configuration and prove Proposition 1.1

As above, let P be the Delzant polytope corresponding to M , and let $f : \mathbb{R}^m \rightarrow \mathbb{R}$ be the convex, rational piecewise-linear function,

$$f = \max\{\lambda_1, \dots, \lambda_p\}, \quad (34)$$

where the λ_j are affine-linear functions with rational coefficients.

Fix an integer R such that $f \geq R$ on \bar{P} and following [D1], §4.2, we define a new polytope

$$Q = Q_{f,R} \subset \mathbb{R}^{m+1}, Q = \{(x, t) : x \in P, 0 < t < R - f(x)\}. \quad (35)$$

By taking a multiple dQ it may be assumed that Q is defined by integral equations. Then dQ is a Delzant polytope of dimension $m+1$ and corresponds to a toric variety W of dimension $m+1$ and a line bundle $\mathcal{L} \rightarrow W$. When $t = 0$ we obtain a natural embedding $\iota : (M, L^d) \rightarrow (W, \mathcal{L})$. Intuitively, the toric degeneration is the singular toric variety corresponding to the

‘top’ of the polytope kdQ . It has p components, one component for each facet of the top or equivalently for each of the affine functions λ_j defining f . More precisely,

PROPOSITION 3.1. (*cf.* [D1] *Proposition 4.2.1*): *There exists a \mathbb{C}^* -equivariant map $p : W \rightarrow \mathbb{CP}^1$ so that $p^{-1}(\infty) = \iota(M)$ such that p restricted to $W \setminus \iota(M)$ is a test configuration for (M, L) with Futaki invariant*

$$F_1 = -\frac{1}{2\text{Vol}(P)} \left(\int_{\partial P} f d\sigma - \alpha \int_P f d\mu \right), \quad \alpha = \frac{\text{Vol}(\partial P)}{\text{Vol}(P)}.$$

The map p is defined as follows: For any j , the ratios $\frac{s_{\alpha,j}}{s_{\alpha,j+1}}$ define \mathbb{C}^* equivariant meromorphic functions on W . In fact, up to scale all of these meromorphic functions agree. Hence we may define p as the common value of the ratios. The map is defined outside the common zeros of $s_{\alpha,j}, s_{\alpha,j+1}$. The sections for $j > 0$ all vanish on $\iota(M)$, so p maps $\iota(M)$ to ∞ .

The fibers $p^{-1}(t)$ are toric varieties isomorphic to M . The central fiber is $p^{-1}(0) = M_0$. Then by definition, M_0 is the zero locus of a holomorphic section σ_0 of $p^*(\mathcal{O}(1))$. By using the exact sequences

$$0 \rightarrow H^0(W, \mathcal{L}^k(-1)) \rightarrow H^0(W, \mathcal{L}^k) \rightarrow H^0(M_0, \mathcal{L}^k) \rightarrow 0$$

and

$$0 \rightarrow H^0(W, \mathcal{L}^k(-1)) \rightarrow H^0(W, \mathcal{L}^k) \rightarrow H^0(\iota(M), \mathcal{L}^k) \rightarrow 0$$

defined by multiplication by σ_0 , resp. σ_1 , one finds that

$$\dim H^0(M_0, \mathcal{L}^k) = \dim H^0(M, \mathcal{L}^k).$$

The principal fact we need about toric test configurations is the following Proposition, which is implicit in [ZZ] (Proposition 3.1):

PROPOSITION 3.2. *The weights of the \mathbb{C}^* action on the spaces $H^0(M_0, \mathcal{L}^k)$ are given by*

$$\eta_\alpha = kd \left(R - f\left(\frac{\alpha}{kd}\right) \right), \quad \alpha \in kdP.$$

Proof. The monomial basis of $H^0(W, \mathcal{L}^k)$ corresponds to lattice points in the associated lattice polytope kdQ . The base of this polytope is thus kdP and the height over a point x is $kd(R - f(\frac{x}{kd}))$. The space $H^0(M_0, \mathcal{L}^k|_{M_0})$ is thus spanned by the monomials

$$z^\alpha w^{kd(R - f(\frac{\alpha}{kd}))}$$

where $\alpha \in kdP$. The \mathbb{C}^* action whose weights we are calculating corresponds to the standard action in the w coordinate and clearly produces the stated weights. \square

The following Corollary immediately implies Proposition 1.1.

COROLLARY 3.3. *The eigenvalues (normalized weights) $\lambda_{\alpha,k}$ are given by*

$$\lambda_{\alpha,k} = kd(R - f\left(\frac{\alpha}{kd}\right)) - \frac{1}{d_k} \sum_{\alpha \in kdP} kd(R - f\left(\frac{\alpha}{kd}\right)). \quad (36)$$

4. THE MEASURES $d\mu_k^z$

In this section we discuss the measures $d\mu_k^z$ (4). Our first purpose is to give the precise statement of Theorem 1.3, and to recall the relevant definitions from the theory of large deviations [dH, DZ]. We then prove Proposition 1.2 using Bergman kernels and Bernstein polynomials. Without loss of generality, we assume that $d = 1$ to simplify the calculation.

A function $I : E \rightarrow [0, \infty]$ is called a rate function if it is proper and lower semicontinuous. A sequence μ_k ($k = 1, 2, \dots$) of sequence of probability measures on a space E is said to satisfy the *large deviation principle with the rate function I* (and with the speed k) if the following conditions are satisfied:

- (1) The level set $I^{-1}[0, c]$ is compact for every $c \in \mathbb{R}$.
- (2) For each closed set F in E , $\limsup_{k \rightarrow \infty} \frac{1}{k} \log \mu_k(F) \leq -\inf_{x \in F} I(x)$.
- (3) For each open set U in E , $\liminf_{k \rightarrow \infty} \frac{1}{k} \log \mu_k(U) \geq -\inf_{x \in U} I(x)$.

Heuristically, in the sense of logarithmic asymptotics, the measure μ_k is a kind of integral of $e^{-kI(x)}$ over the set.

In [DE], Dupuis-Ellis gave an alternative definition in terms of Laplace type integrals and in particular gave a definition of uniform large deviations which is very suitable for our problem. We will state it only in our setting, where the parameter space is the compact toric variety M . Put:

$$F(z, h) = -\inf_{x \in P} (h(x) + I^z(x)). \quad (37)$$

Then $d\mu_k^z$ satisfies the *Laplace principle on P with rate function I^z uniformly on M* if, for all compact subsets $K \subset M$ and all $h \in C_b(P)$ we have:

- (1) For all $c \in \mathbb{R}$, $\bigcup_{z \in M} (I^z)^{-1}[0, c]$ is compact for every $c \in \mathbb{R}$.
- (2) For each $h \in C_b(P)$, $\limsup_{k \rightarrow \infty} \sup_{z \in M} \left(\frac{1}{k} \log \int_P e^{-kh} d\mu_k^z - F(z, h) \right) \leq 0$.
- (3) For each $h \in C_b(P)$, $\liminf_{k \rightarrow \infty} \inf_{z \in M} \left(\frac{1}{k} \log \int_P e^{-kh(x)} d\mu_k^z(x) - F(z, h) \right) \geq 0$.

The upper and lower bounds of course imply, for each $h \in C_b(P)$,

$$\lim_{k \rightarrow \infty} \sup_{z \in M} \left(\frac{1}{k} \log \int_P e^{-kh(x)} d\mu_k^z(x) - F(z, h) \right) = 0.$$

The probability measures of concern in this article are the measures (4), which we often write in the form

$$\mu_k^z = \frac{1}{\Pi_{h_0^k}(z, z)} \sum_{\alpha \in kP} \mathcal{P}_{h^k}(\alpha, z) \delta_{\frac{\alpha}{k}}, \quad (38)$$

where $\mathcal{P}_{h^k}(\alpha, z)$ is given in (17). It is simple to see that μ_k^z is a probability measure since the total mass of the numerator equals $\Pi_{h^k}(z, z)$.

We note that the formula for μ_k^z simplifies when $z \in \mathcal{D}$:

PROPOSITION 4.1. *If $\mu(z) \in F$ where F is a face of P , then*

$$\mu_k^z = \frac{1}{\Pi_{h_0^k}(z, z)} \sum_{\alpha \in kF} \mathcal{P}_{h^k}(\alpha, z) \delta_{\frac{\alpha}{k}}.$$

In the extreme case where z is a fixed point of the \mathbf{T}^m action and $\mu_0(z)$ is a vertex, the measures μ_k^z always equal $\delta_{\mu_0(z)}$.

Proof. It follows easily from [STZ] that $s_\alpha(z) = 0$ for all α such that $\alpha \notin \bar{F}$. In the extreme case where $\mu(z)$ is a vertex v , the only monomial which does not vanish at z is the monomial corresponding to the vertex. We then have

$$\mu_k^z = \frac{1}{\mathcal{P}_{h^k}(v, z)} \mathcal{P}_{h^k}(v, z) \delta_v = \delta_v.$$

□

4.1. Bernstein polynomials and Proposition 1.2. In this section, we prove Proposition 1.2 as an application of Bernstein polynomials in the sense of [Ze2]. We recall here that the k th Bernstein polynomial approximation to $f \in C(\bar{P})$ was defined in [Ze2] by the formula,

$$B_{h^k}(f)(x) := \frac{1}{\Pi_{h^k}(z, z)} \sum_{\alpha \in kP} f\left(\frac{\alpha}{k}\right) \mathcal{P}_{h^k}(\alpha, z), \quad x = \mu(z). \quad (39)$$

The definition extends to characteristic functions of Borel sets $A \subset \bar{P}$ by

$$\mu_k^z(A) = B_k(\chi_A)(x) = \sum_{\alpha \in kP} \chi_A\left(\frac{\alpha}{k}\right) \mathcal{P}_{h^k}(\alpha, z), \quad x = \mu(z).$$

The proof of Proposition 1.2 is as follows:

Proof. For any $f \in C(\bar{P})$,

$$\int f d\mu_k^z = \sum_{\alpha \in kP} f\left(\frac{\alpha}{k}\right) \mathcal{P}_{h^k}(\alpha, z)$$

and the latter is precisely the Bernstein polynomial $B_k(f)(x)$. In [Ze2] it is proved to tend uniformly to $f(z)$. □

We pause to relate Theorem 1.3 to prior results on Bernstein polynomials for characteristic functions. In dimension one, it is a classical result (due to Herzog-Hill) that at a jump discontinuity, the Bernstein polynomials tend to the mean value of the jump. If A is an open set, they converge uniformly to 1 on compact subsets of A , and converge uniformly to zero at an exponential rate on the compact subsets of the interior of A^c as $k \rightarrow \infty$. The large deviations result determines the exponential decay rate. Intuitively, $I^z(x)$ defines a kind of distance from x to z using the $(\mathbb{C}^*)^m$ action, and the limit $\inf_{x \in A} I^z$ defines a kind of Agmon distance from z to A . If $A \subset P^\circ$ and $z \in \partial P$ then $B_k(\chi_A)(x) = 0$, and the distance is infinite. In the case where $P = \Sigma_m$, the unit simplex in dimension m , and with h the Fubini-Study metric on \mathbb{CP}^m and $A \subset P$ is a convex sublattice polytope, such Bernstein polynomials were studied under the name of conditional Szegö kernels in [SZ], since

$$B_k(\chi_A)(x) = \Pi_{h^k|kA}(z, z) = \sum_{\alpha \in kA} \mathcal{P}_{h^k}(\alpha, z), \quad x = \mu(z)$$

is the diagonal of the kernel of the orthogonal projection onto the subspace spanned by monomials z^α with $\alpha \in kA$. The exponential decay rate was determined there when $z \in M^\circ$. In subsequent (as yet unpublished) work, Shiffman-Zelditch have extended the results of [SZ] to non-convex subsets as well.

4.2. Outline of the proof of Theorem 1.3. The proof of Theorem 1.3 is based on the proof of the Gärtner-Ellis theorem [DZ, dH] and on the Laplace large deviations principle of [DE]. The key idea of the Gärtner-Ellis theorem is that the rate function should be the Legendre transform of the scaling limit of the logarithmic moment generating function. But a key component, the uniformity in z , is not a standard feature of the proof and indeed the lower bound is non-uniform (and the upper bound is). However, we only need uniformity of the Laplace large deviations principle in the sense of [DE], and this LDP is uniform. Note that the large deviations principle for each fixed z is equivalent to the Laplace principle for each z , but that the uniformity of the limits is different in the two principles. In outline the proof is as follows:

- (1) In §5, we introduce the logarithmic moment generating function Λ_k^z , and in Proposition 5.1 we determine its scaling limit Λ^z .
- (2) We then introduce its Legendre transform $I^z = \Lambda^{z*}$, and in Proposition 5.2 we calculate I^z .
- (3) In §5.2 we prove the large deviations principle of Theorem ?? for fixed z .
- (4) In §6, we use a special analysis of the weights underlying the measure $d\mu_k^z$ given in Lemma 6.2 to prove the uniform Laplace large deviations principle.

Let us also give a heuristic proof for $z \in M^\circ$ before the formal one. Writing $z = e^{\rho/2+i\theta}$, we then have

$$\mathcal{P}_{h^k}(z, \alpha) \sim k^{-m/2} e^{k(\langle \frac{\alpha}{k}, \rho \rangle - u_\varphi(x) - \varphi(z))},$$

where u_φ is the symplectic potential corresponding to the Kähler potential φ . Hence, for any set $A \subset \bar{P}$,

$$\frac{1}{k} \log \mu_k^z(A) = \frac{1}{k} \log \sum_{\alpha \in kA \cap \mathbb{Z}^m} e^{-k(u_\varphi(\frac{\alpha}{k}) + \langle \rho, \alpha \rangle)} - \varphi(z) + O\left(\frac{\log k}{k}\right). \quad (40)$$

Visibly, the rate function I^z is the function on \bar{P} defined by $u_\varphi(x) - \langle \rho, x \rangle + \varphi(z)$. It is a convex proper function of $x \in P$.

If $z \in \mu_0^{-1}(F)$ for some face F , then by Proposition 4.1 the previous argument goes through as long as we restrict all the calculations to $\alpha \in F$. This gives Theorem 1.3 formally in all cases.

5. LARGE DEVIATIONS PRINCIPLE FOR μ_k^z FOR EACH z

In this section we prove the pointwise large deviations principle stated in Theorems 1.3 and ???. We begin by discussing the logarithmic moment generating function and its scaling limit. These are key ingredients in the Gärtner-Ellis theorem, which we use to conclude the proof. Uniformity in z will be considered in the following section.

In this section we consider the moment generating function (with $t \in \mathbb{R}^m$)

$$\begin{aligned} M_{\mu_k^z}(t) : &= \int_P e^{\langle t, x \rangle} d\mu_k^z(x) \\ &= \sum_{\alpha \in kP} e^{\langle \frac{\alpha}{k}, t \rangle} \frac{\mathcal{P}_{h^k}(\alpha, z)}{\Pi_{h^k}(z, z)}. \end{aligned} \quad (41)$$

Clearly, $M_{\mu_k^z}(t)$ is a convex function of t and is a Bernstein polynomial in the sense of [Ze2] for the function $f_t(x) = e^{\langle t, x \rangle}$, and by Proposition 1.2, $M_{\mu_k^z}(t) \rightarrow e^{\langle t, x \rangle}$ uniformly in z and

$t \in [0, L]$ and $k \rightarrow \infty$. However, the relevant limit

$$\Lambda^z(t) := \limsup_{k \rightarrow \infty} \Lambda_k^z(t), \quad \text{with} \quad \Lambda_k^z(t) := \frac{1}{k} \log M_{\mu_k^z}(kt) \quad (42)$$

is the scaling limit of the logarithmic moment generating function.

By a simple and well-known application of Hölder's inequality (see e.g. [E], Proposition IVV.1.1), $\Lambda_k^z(t)$ and $\Lambda^z(t)$ are convex functions on \mathbb{R}^m for every z .

PROPOSITION 5.1. *We have*

$$\sup_{z \in M} |\Lambda_k^z(t) - \Lambda^z(t)| = o(1), \quad \text{uniformly as } k \rightarrow \infty,$$

where $\Lambda^z(t)$ is given as follows:

- For $z = e^{\rho/2+i\theta} \in M_0$, the open orbit, $\Lambda^z(t) = \varphi_{P^o}(z^{(t+\rho)/2+i\theta}) - \varphi_{P^o}(\rho)$. Here, $e^{t/2}z$ denotes the action of the real subgroup \mathbb{R}_+^m on the open orbit, and φ_{P^o} is the \mathbf{T}^m -invariant Kähler potential on the open orbit.
- For $\mu_0(z) \in F$, a face, $\Lambda^z(t) = \varphi_F(e^{(t'+\rho')/2+i\theta}) - \varphi_F(e^{\rho'/2})$, where φ_F is the open orbit invariant Kähler potential for the toric Kähler subvariety defined by F .
- When z is a fixed point, then $\Lambda^z(t) = 0$.

Proof. When z lies in the open orbit, we may write $z = e^{\rho/2+i\theta}$ and $s_\alpha(z) = z^\alpha e$ where e is a \mathbf{T}^m -invariant frame satisfying $\|e\|_{h_0}^2(z) = e^{-\varphi_{P^o}(z)}$. Then,

$$\begin{aligned} M_{\mu_k^z}(kt) &= \sum_{\alpha \in kP} \frac{e^{\langle \alpha, \rho \rangle} e^{\langle t, \alpha \rangle} e^{-k\varphi_{P^o}(z)}}{Q_{h_0^k}(\alpha) \Pi_{h_0^k}(z, z)} \\ &= e^{k(\varphi_{P^o}(e^{t/2}z) - \varphi_{P^o}(z))} \sum_{\alpha \in kP} \frac{e^{\langle t + \rho, \alpha \rangle} e^{-k\varphi_{P^o}(e^{t/2}z)}}{Q_{h_0^k}(\alpha) \Pi_{h_0^k}(z, z)} \\ &= e^{k(\varphi_{P^o}(e^{t/2}z) - \varphi_{P^o}(z))} \frac{\Pi_{h_0^k}(e^{t/2}z, e^{t/2}z)}{\Pi_{h_0^k}(z, z)}. \end{aligned} \quad (43)$$

Here, $e^{t/2}z$ denotes the \mathbb{C}^* -action (restricted to \mathbb{R}_+^m).

It follows that

$$\begin{aligned} \Lambda_k^z(t) &= \varphi_{P^o}(e^{t/2}z) - \varphi_{P^o}(z) + \frac{1}{k} \log \Pi_{h_0^k}(e^{t/2}z, e^{t/2}z) - \frac{1}{k} \log \Pi_{h_0^k}(z, z) \\ &= \varphi_{P^o}(e^{t/2}z) - \varphi_{P^o}(z) + O\left(\frac{\log k}{k}\right), \end{aligned} \quad (44)$$

with remainder uniform in z by (16).

The calculation for other z is similar, using Proposition 4.1 to reduce the \mathbf{T}^m action to the subtoric variety corresponding to F , and replacing φ_{P^o} by the open orbit toric Kähler potential φ_F on the subtoric variety. At the vertex, the sum reduces to the vertex and the logarithmic moment generating function equals zero by Proposition 4.1. Since the remainders all derive from a uniform Bergman-Szegö kernel expansion (16), there is a uniform limit as $k \rightarrow \infty$ for all z . \square

5.1. The Legendre duals Λ_k^{z*} and Λ^{z*} . The Fenchel-Legendre transform of a convex function F is the convex lower semicontinuous convex function defined by

$$F^*(x) = \mathcal{L}F(x) = \sup_{t \in \mathbb{R}^m} \{\langle x, t \rangle - F(t)\}.$$

We are concerned with the convex functions $F(t) = \Lambda_k^z(t)$ and $F(t) = \Lambda^z(t)$.

In the following Proposition, we refer to the *relative interior* $ri(A)$ of a set $A \subset \bar{P}$. By definition, $ri(P) = P^\circ$ and for a face F , $ri(F) = F^\circ$, the interior of F viewed as a convex subset of the affine space of the same dimension which it spans.

PROPOSITION 5.2. $\Lambda^{z*}(x) = I^z$ is the convex function on \bar{P} given by the following:

- (1) When $z = e^{\rho/2+i\theta}$ lies in the open orbit, then $\Lambda^{z*}(x) = u_0(x) + \varphi_{P^\circ}(e^{\rho/2+i\theta}) - \langle x, \rho \rangle$ for all $x \in \bar{P}$ and $\mathcal{D}(\Lambda^{z*}) = \bar{P}$;
- (2) When $\mu_0(z)$ lies in a face F , and $z = e^{\rho'/2+i\theta'}$ with respect to orbit coordinates on $\mu_0^{-1}(F)$, then $\Lambda^{z*}(x) = u_F(x) + \varphi_F(e^{\rho'/2+i\theta}) - \langle x', \rho' \rangle$ when $x \in F$ and $\Lambda^{z*}(x) = \infty$ if $x \notin \bar{F}$. Thus, $\mathcal{D}_{\Lambda^{z*}} = \bar{F}$ (cf. (50))
- (3) When $\mu_0(z)$ is a vertex v , then $\Lambda^{z*}(v) = 0$ and $\Lambda^{z*}(x) = \infty$ if $x \neq v$ and $\mathcal{D}_{\Lambda^{z*}} = \{v\}$.
- (4) For each $z \in M$, and for any $x \in ri\mathcal{D}(\Lambda^{z*})$, there exists $t = t_*(x, z) \in \mathbb{R}^m$ so that $\nabla_t I^z(t) = x$.

Proof. (1) If $z = e^{\rho/2+i\theta}$ lies in the open orbit,

$$\Lambda^{z*}(x) = \sup_{t \in \mathbb{R}^m} (\langle x, t \rangle - \varphi_{P^\circ}(e^{t/2}z)) + \varphi_{P^\circ}(z). \quad (45)$$

We observe that $\langle x, t \rangle - \varphi_{P^\circ}(e^{t/2}z)$ is concave in t and that

$$\nabla_t \varphi_{P^\circ}(e^{t/2}z) = \nabla_\rho \varphi_{P^\circ}(e^{t/2}z) = \mu_0(e^{t/2}z).$$

The supremum in (45) can only be achieved at the unique critical point $t = t_*(x, z)$ such that

$$x = \mu_0(e^{t/2}z) \iff e^{t/2}z = \mu_0^{-1}(x).$$

We note that $e^{t/2}z$ lies in the open orbit. If $x \in P^\circ$, then there exists a unique $t = t_*(x, z)$ (denoted $\tau_x^P(z)$ in [STZ]) such that $\mu_0^{-1}(x) = e^{t/2}z$, given by

$$t_*(x, z) = \log \mu_0^{-1}(x) - \log |z|. \quad (46)$$

In this case, we have

$$\begin{aligned} \Lambda^{z*}(x) &= \langle x, t_*(x, z) \rangle - \varphi_{P^\circ}(e^{t_*(x,z)/2}z) + \varphi_{P^\circ}(z) \\ &= \langle x, \log \mu_0^{-1}(x) \rangle - \varphi_{P^\circ}(2 \log \mu_0^{-1}(x)) - \langle x, \rho \rangle + \varphi_{P^\circ}(z) \\ &= u_0(x) + \varphi_{P^\circ}(z) - \langle x, \rho \rangle. \end{aligned}$$

This proves (1) when $x \in P^\circ$.

Now consider the case where $z \in M^\circ$ and $x \in \partial P$. We note that $L^z(x) := u_0(x) - \langle x, \rho \rangle + \varphi_{P^\circ}(z)$ is a continuous function of $x \in \bar{P}$. We claim that $\Lambda^{z*}(x)$ is continuous, i.e. it continues to equal this function when $x \in \partial P$, where $u_0(x) = 0$. Since the closure of the open orbit is all of M , there exists a one parameter subgroup $\tau\omega$ with $|\omega| = 1, \tau \in \mathbb{R}$ so that $\lim_{\tau \rightarrow \infty} \mu_0(e^{\tau\omega}z) = x$.

Since $\Lambda^{z*}(x)$ is lower semicontinuous, we automatically have

$$\Lambda^{z*}(x) \leq \liminf_{\tau \rightarrow \infty} \Lambda^{z*}(\mu_0(e^{\tau\omega} z)) = L^z(x).$$

To prove the reverse inequality, we use that

$$\Lambda^{z*}(x) \geq \lim_{\tau \rightarrow \infty} (\langle x, \tau\omega \rangle - (\varphi_{P^o}(e^{\tau\omega/2} z) - \varphi_{P^o}(z))), \quad z = e^{\rho/2+i\theta}.$$

We now claim that

$$(\langle x, \tau\omega \rangle - (\varphi_{P^o}(e^{\tau\omega/2} z) - \varphi_{P^o}(\rho))) - L^z(x) = \langle x, \rho + \tau\omega \rangle - \varphi_{P^o}(e^{\tau\omega/2} z) - u_0(x) \rightarrow 0, \quad \text{as } \tau \rightarrow \infty.$$

Indeed, $u_0(\mu_0(e^{\tau\omega/2} z)) = \langle \mu_0(e^{\tau\omega/2} z), \rho + \tau\omega \rangle - \varphi_{P^o}(e^{\tau\omega/2} z)$, and u_0 is continuous, so the claim reduces to showing that $\langle x - \mu_0(e^{\tau\omega/2} z), \rho + \tau\omega \rangle \rightarrow 0$. Near the boundary, we may approximate the moment map by that of the linear model to check that the expression tends to zero exponentially fast. For instance, in one dimension, with $x = 0$ and $|z| = 1 = -\omega$ the expression becomes $e^{-\tau}(\rho - \tau)$. It follows that $\Lambda^{z*}(x) \geq L^z(x)$. This proves (1) and also (4) when $x \in \bar{P}$ and $z \in M^o$.

(2) Now let us consider the case where $z \in F^o$. We then consider the toric subvariety equal to the closure of $\mu_0^{-1}(F)$ in M . We pick a base point on this toric subvariety and consider orbit coordinates $z' = e^{\rho'/2+i\theta'}$ for the quotient of $\mathbf{T}^m \setminus \mathbf{T}_z^m$ where \mathbf{T}_z^m is the isotropy group of z . Then,

$$\Lambda^{z*}(x) \geq \sup_{t \in \mathbb{R}^m} (\langle x, t \rangle - \varphi_F(t' + \rho')) - \varphi_F(\rho'). \quad (47)$$

Here, we note that $e^t \cdot z = e^{t'} \cdot z$ where $e^{t'}$ is a representative of e^t in $\mathbf{T}^m \setminus \mathbf{T}_z^m$.

As above, we find that the supremum is only achieved when $x \in F$ and then the calculation becomes the open orbit calculation for the sub-toric variety. By the previous argument, the open orbit formula extends to the closure of $\mu_0^{-1}(F)$ by continuity and in addition (4) holds for $x \in F^o$.

On the other hand, if $x \notin F$ then the supremum is not achieved. Write $x = (x', x'')$ and similarly for t . Then (47) is the sum of $\langle x'', t'' \rangle$ plus terms depending only on t' . If $x'' \neq 0$ we can let $t'' = rx''$ with $r > 0$ and find that the supremum equals $+\infty$.

(3) In this case, $\Lambda^z = 0$ and it is obvious that the supremum is infinite if $x \neq 0$. Here, the coordinates are chosen so that the vertex occurs at 0. □

Remark: As a check on signs, we note that $\Lambda^{z*}(x)$ should be non-negative and convex. Indeed, $\Lambda^{z*}(x)$ is convex and takes its minimal value of 0 at $\mu(z) = x$.

We end this section with the following important ingredient in the uniform estimates:

PROPOSITION 5.3. *We have*

$$\Lambda_k^{z*}(t) = \Lambda^{z*}(t) + O\left(\frac{\log k}{k}\right), \quad \Lambda_{k\delta}^{z*}(t) = \Lambda_{\delta}^{z*}(t) + O\left(\frac{\log k}{k}\right)$$

where the remainder is uniform in z, t (and δ).

Proof. By (44) and by (16), we have

$$\begin{aligned}\Lambda_k^{z*}(x) &= \sup_{t \in \mathbb{R}^m} \left(\langle t, x \rangle - \varphi_{P^o}(e^{t/2}z) - \frac{1}{k} \log \Pi_{h_0^k}(e^{t/2}z, e^{t/2}z) \right) + \varphi_{P^o}(z) + \frac{1}{k} \log \Pi_{h_0^k}(z, z) \\ &= \sup_{t \in \mathbb{R}^m} \left(\langle t, x \rangle - \varphi_{P^o}(e^{t/2}z) \right) + \varphi_{P^o}(z) + O\left(\frac{\log k}{k}\right).\end{aligned}\tag{48}$$

5.2. Proof of Theorem 1.3: Pointwise large deviations. We now prove that for each z , $d\mu_k^z$ satisfies the large deviations principle with rate function

$$I^z = \Lambda^{z*}.\tag{49}$$

We use the two notations interchangeably in what follows. The proof is an application of the Gärtner-Ellis theorem as in [DZ, dH, E]. We postpone a discussion of uniformity to the next section. For the sake of completeness we recall the statement of the Gärtner-Ellis theorem:

Let μ_k be a sequence of probability measures on \mathbb{R}^n . Assume that

- (1) *the scaled logarithmic moment generating function $\Lambda(t)$ of $\{d\mu_k\}$ exists and that 0 lies in its domain $\mathcal{D}(\Lambda)$;*
- (2) *Λ is lower-semicontinuous and differentiable on \mathbb{R}^n .*

Then μ_k have an LDP with speed k and rate function Λ^ .* Here, the domain of a convex function F is defined by

$$\mathcal{D}_\Lambda = \{t \in \mathbb{R}^m : F(t) < \infty\}.\tag{50}$$

Proof of the large deviations statement: It suffices to verify that the hypotheses of the Gärtner-Ellis theorem are satisfied. All the work has been done in the previous section.

It follows from Proposition 5.1 that $\Lambda^z(t)$ satisfies the assumption (1) of the Gärtner-Ellis theorem: the limit (42) exists for all t , and the origin belongs to its domain \mathcal{D}_{Λ^z} . Indeed, for all $z \in M$, $\mathcal{D}_{\Lambda^z} = \mathbb{R}^m$.

Further, Λ^z is differentiable everywhere for every z with

$$\nabla_t \Lambda^z(t) = \nabla \varphi_{P^o}(e^{t/2}z) = \mu_0(e^{t/2}z).\tag{51}$$

It follows that μ_k^z satisfy for each z the LDP with speed k and rate function $I^z = \Lambda^{z*}$. □

6. UNIFORM LAPLACE PRINCIPLE AND UNIFORM VARADHAN'S LEMMA

We now consider uniformity of the large deviations principle and in particular, the key issue of uniformity of Varadhan's Lemma. Our goal is to prove the first part of Theorem 1.4, which we restate for clarity:

THEOREM 6.1. *For any $L > 0$, $\sup_{(z,t) \in M \times [0,L]} |\psi_k(t, z) - \psi_t(z)| = 0$.*

In view of Proposition 1.1, we need to understand the convergence of the sequence of relative Kähler potentials (5), of which the key issue is the convergence of

$$\tilde{\psi}_k(z, t) := \frac{1}{k} \log \int_P e^{kt(R-f(x))} d\mu_k^z(x).\tag{52}$$

We prove the C^0 convergence of $\psi_k \rightarrow \psi$ as an application of Varadhan's Lemma to (52) (cf. [dH], Theorem III. 13). We recall the statement of the Lemma:

Varadhan's Lemma *Let $d\mu_k$ be probability measures on X which satisfy the LDP with rate k and rate function I on X . Let F be a continuous function on X which is bounded from above. Then*

$$\lim_{k \rightarrow \infty} \frac{1}{k} \log \int_X e^{kF(x)} d\mu_k(x) = \sup_{x \in X} [F(x) - I(x)].$$

It follows immediately from Varadhan's Lemma and the pointwise large deviations result of §5 that $\psi_k(t, z) \rightarrow \psi(t, z)$ pointwise for each (t, z) . However we would like to prove uniform convergence for $t \in [0, L]$ and $z \in M$.

In proving C^0 convergence, we do not use any special properties of $t(R - f)$ beyond the fact that it is a continuous function on the closed polytope P . Hence, our uniform convergence proof automatically implies the uniform Laplace principle stated at the beginning of §4.

6.1. Weights and rates. □

The following Lemma reflects the fact that the weights of our special measure are already very close to the rate function:

LEMMA 6.2. *Let $L_k(z, \frac{\alpha}{k}) = \frac{1}{k} \log \frac{|s_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} + I^z(\frac{\alpha}{k})$. Then $L_k(z, \frac{\alpha}{k}) = -\frac{1}{k} \log Q_{h^k}(\alpha) + u(\frac{\alpha}{k})$ for $z \in M^o$ and satisfies*

$$L_k(z, \frac{\alpha}{k}) = O(\frac{1}{k}), \quad (k \rightarrow \infty)$$

uniformly in $z \in M^o$ and $\alpha \in kP$. The same formula and uniform asymptotics hold when $\frac{\alpha}{k} \in F$ and $z \in \mu^{-1}(F)$.

Proof. First assume that $z \in M^o$. Then,

$$\frac{1}{k} \log \frac{|s_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} = \langle \frac{\alpha}{k}, \rho \rangle - \varphi_{P^o}(\rho) - \frac{1}{k} \log Q_{h^k}(\alpha), \quad (53)$$

while

$$I^z(\frac{\alpha}{k}) = -\langle \frac{\alpha}{k}, \rho \rangle + u_0(\frac{\alpha}{k}) + \varphi_{P^o}(\rho). \quad (54)$$

Hence,

$$\frac{1}{k} \log \frac{|s_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} + I^z(\frac{\alpha}{k}) = -\frac{1}{k} \log Q_{h^k}(\alpha) + u_0(\frac{\alpha}{k}). \quad (55)$$

We observe that the right side is independent of z and extends continuously from $\frac{1}{k}$ -lattice points $\frac{\alpha}{k}$ to general $x \in P$, proving the first statement of the Proposition.

Now suppose that $z \in F$, a face of ∂P . Then $s_\alpha(z) = 0$ and $\log \frac{|s_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} = -\infty$ unless $\frac{\alpha}{k} \in \overline{F}$. Also, $I^z(\frac{\alpha}{k}) = +\infty$ when $\frac{\alpha}{k} \notin \overline{F}$. The formula above gives a meaning to $L_k(z, \frac{\alpha}{k})$ in this case. When $z \in F$ and $\frac{\alpha}{k} \in \overline{F}$ then in slice-orbit coordinates for F ,

$$\frac{1}{k} \log \frac{|s_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} = \langle \frac{\alpha''}{k}, \rho'' \rangle - \varphi_F(\rho'') - \frac{1}{k} \log Q_{h^k}(\alpha),$$

while

$$I^z(\frac{\alpha}{k}) = -\langle \frac{\alpha''}{k}, \rho'' \rangle + u_0(\frac{\alpha}{k}) + \varphi_F(\rho'').$$

Thus, the same formula holds in the case $z \in \mathcal{D}$. Indeed, the extension of $L_k(z, \frac{\alpha}{k})$ by constancy to $M \times P$ is consistent with its extension using the definition of I^z . The estimate (25) then completes the proof. \square

6.2. Uniform large deviations upper bound. In this section we prove that the large deviations upper bounds are uniform.

PROPOSITION 6.3. *For any compact subset $K \subset \bar{P}$, we have the uniform upper bound*

$$\frac{1}{k} \log \mu_k^z(K) \leq - \inf_{x \in K} I^z(x) + O\left(\frac{\log k}{k}\right),$$

where the remainder is uniform in z .

Proof. By definition,

$$\frac{1}{k} \log \mu_k^z(K) = \frac{1}{k} \log \sum_{\alpha \in kP: \frac{\alpha}{k} \in K} \frac{\mathcal{P}_{h^k}(\alpha, z)}{\Pi_{h^k}(z, z)}.$$

By Proposition 4.1, if $z \in \mathcal{D}$ then μ_k^z is supported on the face of $\mu(z)$. Hence for any z , Lemma 6.2 applies to all terms in the sum for μ_k^z and termwise we have

$$\mathcal{P}_{h^k}(\alpha, z) = e^{-kI^z(\frac{\alpha}{k})} e^{O(1)},$$

where $O(1)$ is uniform in z, α . Since $\log \Pi_{h^k}(z, z) = O(\log k)$ uniformly in z , it follows that

$$\frac{1}{k} \log \mu_k^z(K) = \frac{1}{k} \log \sum_{\alpha \in kP: \frac{\alpha}{k} \in K} e^{-kI^z(\frac{\alpha}{k})} + O\left(\frac{1}{k}\right) + O\left(\frac{\log k}{k}\right)$$

The Proposition then follows from the facts that $e^{-kI^z(\frac{\alpha}{k})} \leq e^{-k \inf_{x \in K} I^z(x)}$ for every term in the sum, and that the number of terms in the sum is $O(k^m)$. \square

6.3. Lower bound. Unfortunately, the lower bound in the large deviations principle for open sets is not uniform in z . However, for Varadhan's Lemma, we only need a special case. The following gives an example of it, although we will need to generalize it later. For $z \in M, x_0 \in P$, and $\epsilon > 0$, put

$$U(z, x_0, \epsilon) = \{x \in P : I^z(x) < I^z(x_0) + \epsilon\}. \quad (56)$$

PROPOSITION 6.4. *We have*

$$\frac{1}{k} \log \mu_k^z(U(z, x_0, \epsilon)) \geq -I^z(x_0) - \epsilon + o(1), \quad (57)$$

where the remainder is uniform in z and tends to zero as $k \rightarrow \infty$.

Proof. First, $U(z, x_0, \epsilon) \cap \frac{1}{k}\mathbb{Z}^m \neq \emptyset$ and indeed, there exists $\alpha_k(z) \in U(z, x_0, \epsilon) \cap \frac{1}{k}\mathbb{Z}^m$ satisfying:

- (1) $|x_0 - \alpha_k(z)| \leq \frac{\sqrt{m}}{k}$;
- (2) When $z \in M^o$, $\langle \alpha_k(z) - x_0, \log |z| \rangle \geq 0$. When $z \in \mu^{-1}(F)$ then $\langle \alpha'_k(z) - x_0, \log |z'| \rangle \geq 0$.

As in the upper bound,

$$\begin{aligned}
\frac{1}{k} \log \mu_k^z(U(z, x_0, \epsilon)) &= \frac{1}{k} \log \sum_{\alpha \in kP: I^z(\frac{\alpha}{k}) < I^z(x_0) + \epsilon} \frac{\mathcal{P}_{h^k}(\alpha, z)}{\Pi_{h^k}(z, z)} \\
&= \frac{1}{k} \log \sum_{\alpha \in kP: I^z(\frac{\alpha}{k}) < I^z(x_0) + \epsilon} e^{-kI^z(\frac{\alpha}{k})} + O(\frac{1}{k}) + O(\frac{\log k}{k}) \\
&\geq \frac{1}{k} \log \sum_{\alpha \in kP: I^z(\frac{\alpha}{k}) < I^z(x_0) + \epsilon} e^{-k(I^z(x_0) + \epsilon)} + O(\frac{1}{k}) + O(\frac{\log k}{k}) \\
&\geq -I^z(x_0) + \epsilon + O(\frac{1}{k}) + O(\frac{\log k}{k}).
\end{aligned} \tag{58}$$

In the last inequality, we used that

$$I^z(\alpha_k) - I^z(x_0) = -\langle \alpha_x - x_0, \log |z|^2 \rangle + u_0(x) - u_0(\alpha_k),$$

and used the continuity of u_0 to see that $|I^z(\alpha_k) - I^z(x_0)| \leq o(1)$ as $k \rightarrow \infty$. \square

6.4. Proof of Theorem 6.1. We now prove Theorem 6.1. We employ the notation F_t as in (6). We derive it from the more general uniform Laplace large deviations principle: for each $h \in C_b(P)$,

$$\lim_{k \rightarrow \infty} \sup_{z \in M} \left(\frac{1}{k} \log \int_P e^{-kh(x)} d\mu_k^z(x) - F(z, h) \right) = 0,$$

where $F(z, h)$ is defined in (37). Theorem 6.1 is the special case with $h = -F_t$.

Proof. We begin with the uniform lower bound, first proving a generalization of Proposition 6.4:

LEMMA 6.5. *Let $h \in C(P)$. Then,*

$$\frac{1}{k} \log \sum_{\alpha \in kP: I^z(\frac{\alpha}{k}) + h(\frac{\alpha}{k}) < I^z(x_0) + h(x_0) + \epsilon} e^{-kh(\frac{\alpha}{k})} \frac{\mathcal{P}_{h^k}(\alpha, z)}{\Pi_{h^k}(z, z)} \geq -I^z(x_0) - h(x_0) + \epsilon + o(1), \tag{59}$$

where the remainder is uniform in z and x_0 and tends to zero as $k \rightarrow \infty$.

Proof. With no loss of generality we may assume $h \geq 0$, i.e. $e^{-kh(x)} \leq 1$; if not, we may replace h by $h - \min h$. Then the left side is

$$\begin{aligned}
&= \frac{1}{k} \log \sum_{\alpha \in kP: I^z(\frac{\alpha}{k}) + h(\frac{\alpha}{k}) < I^z(x_0) + h(x_0) + \epsilon} e^{-kh(\frac{\alpha}{k})} e^{-kI^z(\frac{\alpha}{k})} + O(\frac{1}{k}) + O(\frac{\log k}{k}) \\
&\geq \frac{1}{k} \log \sum_{\alpha \in kP: I^z(\frac{\alpha}{k}) + h(\frac{\alpha}{k}) < I^z(x_0) + h(x_0) + \epsilon} e^{-k(I^z(x_0) + h(x_0) + \epsilon)} + O(\frac{1}{k}) + O(\frac{\log k}{k}) \\
&\geq -I^z(x_0) - h(x_0) + \epsilon + O(\frac{1}{k}) + O(\frac{\log k}{k}).
\end{aligned} \tag{60}$$

\square

Since x_0, ϵ are arbitrary, it follows that

$$\frac{1}{k} \log \int_P e^{-kh(x)} d\mu_k^z(x) \geq \inf_{x \in P} (-h(x) - I^z(x)) + o(1), \tag{61}$$

giving a uniform lower bound.

For the upper bound, we use that

$$\begin{aligned}
\frac{1}{k} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} e^{kh(\frac{\alpha}{k})} \frac{\mathcal{P}_{h^k}(\alpha, z)}{\Pi_{h^k}(z, z)} &= \frac{1}{k} \log \sum_{\alpha \in kP} e^{-kh(\frac{\alpha}{k})} e^{-kI^z(\frac{\alpha}{k})} + O(\frac{1}{k}) + O(\frac{\log k}{k}) \\
&\leq \max_{\alpha \in kP \cap \mathbb{Z}^m} - (h(\frac{\alpha}{k}) + I^z(\frac{\alpha}{k})) \\
&\leq \max_{x \in P} - (h(x) + I^z(x)).
\end{aligned} \tag{62}$$

□

This completes the proof of uniform Laplace large deviations principle and also of Theorem 6.1.

7. MOMENT MAP μ_t AND SUBDIFFERENTIAL OF $u + tf$

We recall that the moment map $\mu = \mu_\omega$ of a smooth toric Kähler variety is defined by $\mu_\omega(z) = (H_1, \dots, H_m) \in P \subset \mathbb{R}^m$, with $dH_j = \omega(\frac{\partial}{\partial \theta_j}, \cdot)$ where $\{\frac{\partial}{\partial \theta_j}\}$ is a basis for $\text{Lie}(\mathbf{T}^m)$. On the open orbit it is given by $\mu_\omega(e^{\rho/2}) = \nabla_\rho \varphi(\rho)$ where φ is the open orbit Kähler potential. On a non-open orbit $\mu_\omega^{-1}(F)$ for some face F , it is given by the analogous formula but with φ_F replacing φ and the ρ'' orbit coordinates of §2 replacing ρ . Thus, the moment map defines a stratified Lagrangian torus fibration $\mu : M \rightarrow P$ which is a fibering $M^o \rightarrow P^o$ away from the divisor at infinity and boundary of P . In the real picture, if we divide by \mathbf{T}^m , the moment map on the quotient, $\mu_\omega : M/\mathbf{T}^m \rightarrow P$ is a diffeomorphism of manifolds with corners.

Our first purpose in this section is to prove that the interior and face-wise gradient maps are well-defined for the singular potentials ψ_t , and define a moment map μ_t for the singular form ω_t (cf. (8)). We then study the regularity and mapping properties of μ_t . We would like to generalize the homeomorphism property $\mu_\omega : M/\mathbf{T}^m \rightarrow P$ to the singular Kähler forms ω_t . To do so, we observe that the diffeomorphism from $M^o/\mathbf{T}^m \rightarrow P^o$ is the standard inverse relation between the gradient map $\nabla \varphi_{P^o}$ defined by the Kähler potential and the inverse gradient map ∇u_0 defined by its Legendre transform, the symplectic potential. For the singular metrics ω_t , the moment maps μ_t is no longer a homeomorphism of this kind. Indeed, the gradient $\nabla(u_0 + tf)$ of its symplectic potential is not differentiable on its codimension-one corner set

$$\mathcal{C} = \{x \in \bar{P}, \exists i \neq j : \lambda_i(x) = \lambda_j(x)\}.$$

The derivative is discontinuous and its image disconnects the complementary regions. We write $P \setminus \mathcal{C} = \bigcup_{j=1}^R P_j$, where $u_0 + tf = u_0 + t(\langle \lambda_j, x \rangle + \eta_j)$ and refer to P_j as the j th smooth chamber.

However, the inverse relation between μ_t and $\partial(u_0 + tf)$ can be re-instated as a homeomorphism if we replace the gradient map $\nabla(u_0 + tf)$ by the set-valued subdifferential $\partial(u_0 + tf)$ and P by its graph $\mathcal{G}\partial(u_0 + tf)$. This explains why ψ_t can be C^1 although $u_0 + tf$ is not.

At the boundary ∂P , ∇u has logarithmic singularities and hence the graph of ∇u , hence of $\nabla(u_0 + tf)$, is not well-defined. This is because ∇u in the smooth case must invert $\nabla \varphi$ and send points of ∂P to \mathcal{D} . The polytope P already compactifies this picture, and this suggests that we replace the graph of the subdifferential $\partial(u_0 + tf)$ by the graph $\mathcal{G}\partial(tf)$ of the subdifferential of the relative symplectic potential, i.e. $\partial(tf)$ over P . It is of course homeomorphic to the graph of $\nabla(u_0 + tf)$ away from ∂P and compactifies the graph on ∂P .

We therefore prove:

THEOREM 7.1. *μ_t is Lipschitz continuous on M for each $t \geq 0$. Moreover, μ_t has a natural lift $\tilde{\mu}_t : M \rightarrow t\mathcal{G}\partial f$ which is a homeomorphism (see Definition 7.4).*

7.1. Regularity of ψ_t and definition of μ_t on orbits. Using standard results of convex analysis, we can immediately obtain the regularity of ψ_t on the open orbit and along any other orbit.

Let us first recall the relation between the relative Kähler potential ψ_t and the open orbit absolute Kähler potential

PROPOSITION 7.2. *For $t \geq 0$,*

- (1) $\mathcal{L}(u_0 + tf) \in C^1(\mathbb{R}^m)$. Hence $\psi_t|_{M^o} \in C^1(M^o)$.
- (2) For any face F of P , $\mathcal{L}_F(u + tf) \in C^1(\mathbb{R}^{m-k})$. Hence $\psi_t|_{M_F} \in C^1(M_F)$.

Proof. The first statement follows immediately from Theorem 26.3 of [R]: *A closed proper convex function is essentially strictly convex if and only if its Legendre conjugate is essentially smooth.* It suffices to recall the definitions and to verify that $u + tf$ is essentially strictly convex on P^o .

A proper convex function g is called *essentially strictly convex* (see [R], page 253) if g is strictly convex on every convex subset of $\text{dom}(\partial g) = \{x : \partial g(x) \neq \emptyset\}$. Here, $\partial g(x)$ is the subdifferential of g at x . This property is satisfied by $g = u + tf$ since u is strictly convex on \bar{P} and since f is convex so that $u + tf$ is strictly convex for $t > 0$. It follows that $\mathcal{L}(u + tf)$ is essentially smooth.

We recall that g is *essentially smooth* ([R], page 251) if its domain $\text{dom}(g)$ has non-empty interior C , if g is differentiable on all of C and if $|\nabla g(x_j)| \rightarrow \infty$ when $x_j \rightarrow \partial C$ (the last condition is vacuous if $C = \mathbb{R}^m$.) Thus, $\mathcal{L}(u + tf)$ is differentiable on \mathbb{R}^m . To complete the proof, it suffices to recall that an everywhere differentiable convex function is automatically C^1 (cf. [R], Corollary 25.5.1).

The same proof shows that the restrictions of ψ_t to the sub-toric varieties M_F of ψ_t are also C^1 along the sub-toric varieties, proving the second statement. \square

COROLLARY 7.3. *The gradient maps*

- $\nabla_\rho \varphi_t(\rho) : \mathbb{R}^m \rightarrow P^o$,
- $\nabla_{\rho''} \varphi_t|_{M_F}(\rho'') : \mathbb{R}^{m-k} \rightarrow F$

are well defined and continuous.

The Corollary serves to define the moment map μ_t :

$$\mu_t(z) = \begin{cases} \nabla_\rho(\psi_t + \varphi_{P^o})(\rho), & z = e^{\rho/2+i\theta} \in M_0; \\ \nabla_{\rho''}(\psi_t + \varphi_{P^o})(\rho''), & z = e^{\rho''/2+i\theta''} \in M_F. \end{cases} \quad (63)$$

7.2. Moment map μ_t and subdifferential map. So far, we have shown that μ_t as defined by (63) is continuous in (t, z) on the interior and along each boundary face. To complete the study of μ_t we need to prove the homeomorphism properties and to analyze continuity across \mathcal{D} . We will need to recall some further definitions in convex analysis on \mathbb{R}^m , following [R, HUL]. First, a convex function g is called *proper* if $g(x) < \infty$ for at least one x and

$g(x) > -\infty$ for all x (cf. [R], page 24). When g is a proper convex function, it is *closed* if it is lower semi-continuous ([R], page 54). These conditions are trivially satisfied for $g = u + tf$. A subgradient of g at $x \in \mathbb{R}^m$ is a vector $x^* \in \mathbb{R}^m$ such that

$$g(y) - g(x) \geq \langle x^*, y - x \rangle, \quad \forall y,$$

i.e. if $g(x) + \langle x^*, y - x \rangle$ is a supporting hyperplane to the epi-graph $\text{epi}(g)$ at $(x, g(x))$. The subdifferential of g at point x_0 is the set of subgradients at x , i.e.

$$\partial g(x_0) = \{\rho : g(y) - g(x) \geq \langle \rho, y - x \rangle, \quad \forall y\},$$

A convex function g is differentiable at x if and only if $\partial g(x)$ is a single vector (hence $\nabla g(x)$).

The graph of the subdifferential of g is the set

$$\mathcal{G}(\partial g) = \{(x, \rho) : \rho \in \partial g(x)\} \subset T^*\mathbb{R}^m. \quad (64)$$

In the smooth case, it is a Lagrangian submanifold. In the cornered case it is a Lagrangian submanifold with corners. An illustration in the one-dimensional case may be found [HUL]I (p. 23). As the illustration shows, the derivative has a jump at each corner point x , and the graph fills in the jump at x with a vertical interval $[D_- f(x), D_+ f(x)]$ where $D_{\mp} f(x)$ are the left/right derivatives.

7.3. Legendre transforms, gradient maps and Lagrangian graphs. We recall that the Legendre transform of a convex function g on \mathbb{R}^m is defined by $\mathcal{L}g(x) = g^*(x) = \sup_{\rho \in \mathbb{R}^m} (\langle x, \rho \rangle - g(x))$.

When the gradient map ∇g is invertible, its inverse is the gradient map ∇g^* . Another way to put this is that the graph $gr\nabla = (x, \nabla g(x))$ of ∇g is a Lagrangian submanifold $\Lambda \subset T^*\mathbb{R}^m = \mathbb{R}_{x,\xi}^{2m}$ which projects without singularities to the base \mathbb{R}_x^m . One says that g parameterize Λ . Open sets of Λ which project without singularities to the fiber \mathbb{R}_ξ^m can be parameterized as graphs $(\nabla g^*(\xi), \xi)$ of $\nabla g^* : \mathbb{R}_\xi^m \rightarrow T^*\mathbb{R}^m$. We put $\iota(x, \xi) = (\xi, x)$ so that Λ is (locally) given as $\iota \mathcal{G} \nabla g^*$. One then says that g^* parameterizes Λ (locally). Applying ι is equivalent to the statement that $\nabla g, \nabla g^*$ are inverse maps.

When g is not differentiable, one can replace the graphs of ∇g and ∇g^* by the subdifferentials $\partial g, \partial g^*$. The graph of ∂g is a piecewise smooth Lagrangian manifold Λ with corners over the non-smooth points of g . Λ is also the graph of $\iota \partial g^*$ over the subset of \mathbb{R}_ξ^m where it projects. As a simple example consider $g(x) = |x|$: The graph of ∂g consists of the graph of $\xi = -1$ on \mathbb{R}_- together with the vertical segment $x = 0, \xi \in (-1, 1)$ together with the graph of $\xi = 1$ on \mathbb{R}_+ . We see that the graph projects to the interval $[-1, 1] \subset \mathbb{R}_\xi$. Hence the domain of $\mathcal{L}|x|$ is $[-1, 1]$. Further, $\mathcal{L}|x| = 0$ on $[-1, 1]$ and $\pm\infty$ for $\xi > 1$, resp. $\xi < -1$. Hence, $(\partial \mathcal{L}|x|)(\xi) = 0$ for $\xi \in (-1, 1)$ and it is the half-line $x > 1$ at $\xi = 1$, the half line $x < -1$ at $\xi = -1$ and it is undefined elsewhere. We see that $\iota \mathcal{G} \partial \mathcal{L}|x| = \mathcal{G} \partial |x|$ and hence in a generalized sense the set-valued maps $\partial |x|$ and $\partial \mathcal{L}|x|$ are inverses.

7.4. The moment map and subdifferential of $u + tf$ on the interior. We now return to the moment map μ_t , where the convex function of interest ψ_t (cf. (7)).

When $t = 0$, the moment map $\mu_0 = \nabla \varphi_{P^0}(\rho)$ is a homeomorphism from $M_{\mathbb{R}}^o \rightarrow P^o$ which is inverted by $\exp \frac{1}{2} du_0$ (cf. (22)). In the non-smooth case, the gradient map is undefined at the corner set. We then replace the gradient at the corner by the subdifferential $\partial(u_0 + tf)$ [R, HUL]. The following Proposition identifies this set. Given x , we denote by $J_x = \{j : \lambda_j(x) = \max\{\lambda_1(x), \dots, \lambda_r(x)\}\}$.

PROPOSITION 7.4. *Let $f = \max\{\lambda_1, \dots, \lambda_r\}$ be a piecewise affine function on P . Then for $x \in P^o$,*

$$\partial(u_0 + tf)(x) = \nabla u(x) + tCH\{\nu_j : j \in J_x\}.$$

Proof. Clearly, $\partial(u_0 + tf)(x) = \nabla u_0(x) + t\partial f(x)$, so it suffices to show that $\partial f(x) = CH\{\lambda_j : j \in J_x\}$. But it is well known that the subdifferential of the maximum $g = \max_j g_j$ of r convex functions is the convex hull of the gradients of those g_j such that $g(x) = g_j(x)$ (see e.g. [IT] for much more general results). \square

PROPOSITION 7.5. *The graph of the subdifferential, $\mathcal{G}\partial(u_0 + tf)(x) \subset T^*P^o$, is a piecewise smooth Lagrangian submanifold with corners. The graph of the subdifferential of the relative symplectic, $\mathcal{G}\partial(tf)(x) \subset T^*P$, is a piecewise linear Lagrangian submanifold with corners over P .*

Proof. The term ∇u_0 is clearly irrelevant to the first statement and hence the second statement implies the first.

The statement is local and can be checked in the model examples where the affine functions are the coordinates x_1, \dots, x_p of \mathbb{R}^m . The corner set consists of the large diagonal hyperplanes $x_1 = \dots = x_p = 0$ for some $p \leq m$ and distinct indices among $1, \dots, m$. On the complement of the large diagonal hyperplanes, the subdifferential is a constant vector e_j for some j . Over any point x of the diagonal hyperplanes $x_i = x_j$, the subdifferential contains the one-dimensional convex hull $CH(e_i, e_j)$. This convex hull over the hyperplane is an m dimensional linear manifold bridging the two constant graphs and the union is a piecewise linear Lagrangian submanifold Λ_1 . Over higher s -codimension intersections, the subdifferential contains s such one-dimensional convex sets and equals their convex hull. It follows that the full subdifferential is piecewise linear of dimension m . The canonical symplectic form of T^*M vanishes on all smooth faces and on all vectors tangent to the corners. \square

Although $\partial(u + tf)$ is multi-valued, $\partial\varphi_t(\rho)$ is a singleton for all ρ :

PROPOSITION 7.6. *For each $\rho \in \mathbb{R}^m$, there exists precisely one $x \in P^o$ so that $(x, \rho) \in \mathcal{G}(\partial(u_0 + tf))$.*

Proof. This is actually equivalent to Proposition 7.2 and is the key step in the proof we quoted from Theorem 26.3 of [R]: a convex function f is differentiable at x if and only if ∂f consists of just one vector (i.e. $\nabla f(x)$.)

Let us verify that $\partial(u_0 + tf)(x_1) \cap \partial(u_0 + tf)(x_2) = \emptyset$ when $x_1 \neq x_2$. We argue by contradiction: suppose that $x^* \in \partial(u_0 + tf)(x_1) \cap \partial(u_0 + tf)(x_2)$. Then the graph of $\langle x^*, z \rangle - (u_0 + tf)^*(x^*)$ is a non-vertical supporting hyperplane to $\text{epi}(u_0 + tf)$ containing $(x_1, (u_0 + tf)(x_1))$ and $(x_2, (u_0 + tf)(x_2))$. But then the line segment joining these points lies in the hyperplane, so $(u_0 + tf)$ cannot be strictly convex along the line segment joining x_1 and x_2 . \square

COROLLARY 7.7. *If ψ_t is associated to a non-trivial test configuration, then $\psi_t \notin C^2(M)$.*

Proof. Indeed, $\psi_t|_{M^o} \notin C^2(M^o)$ since $\nabla^2\psi_t$ has a kernel at each point in the image of the subdifferential of the corner set but it is strictly positive definite in the smooth regions. \square

Definition: The lifted interior moment map is the composite map

$$\begin{aligned} \tilde{\mu}_t(\rho) = (\mu_t(\rho), \rho) : M^o &\rightarrow \mathcal{G}(\partial(u_0 + tf)) \subset T^*P^o \\ &\downarrow \\ &\mathcal{G}(\partial(tf)) \subset T^*P^o. \end{aligned} \tag{65}$$

The downwards arrow is the map $(\mu_t(\rho), \rho) \rightarrow (\mu_t(\rho), \rho - \nabla u_0(\mu_t(\rho)))$. We define analogous maps along other orbits where we replace φ_{P^o} by φ_F and u_0 by u_F (the F -Legendre transform of φ_F .)

COROLLARY 7.8. *The ‘lifted moment map’ $\tilde{\mu}_t(\rho) : M_{\mathbb{R}}^o \rightarrow \mathcal{G}(\partial(u + tf))|_{P^o}$ is a homeomorphism. The same is true for restrictions to orbits in \mathcal{D} and the corresponding boundary faces.*

Proof. The graph of $\mu_t = \nabla\psi_t$ is homeomorphic to the graph of $\partial(u_0 + tf)$ under ι , since they are Legendre duals: i.e. $\iota\mathcal{G}\partial\psi_t = \mathcal{G}\partial(u_0 + tf)$. The ‘projection’ to $\mathcal{G}(tf)$ obtained by composing with the subtraction map $\rho - \nabla u_0(\mu_t(\rho))$ is clearly a homeomorphism, hence so is the composition. \square

7.5. Moment map near the divisor at infinity $\mathcal{D} \cap M_{\mathbb{R}}$. So far, we have proved that the lifted moment maps are homeomorphisms from orbits to graphs of the subdifferential of tf on faces of P . Since such orbits (resp. faces) form a partition of M (resp. P), the remaining steps in the proof of Theorem 7.1 are to prove that μ_t is continuous on all of M and has a continuous inverse μ_t^{-1} from P to the closure of the real open orbit.

We study the behavior of μ_t near \mathcal{D} and the boundary behavior of its inverse by exponentiating the subdifferential. To understand this from an invariant viewpoint, we recall that $du(x) \in T^*(\text{Lie}(\mathbf{T}^m)^*) \simeq \text{Lie}(\mathbb{R}_+^m)$, so that we may regard $(x, du_0(x)) \in (\text{Lie}(\mathbf{T}^m)^*) \oplus \text{Lie}(\mathbb{R}_+^m)$. In the same way, we define the graph of the subdifferential by

$$\mathcal{G}(\partial(u_0 + tf)) := \{(x, \rho) \in \text{Lie}(\mathbf{T}^m)^* \oplus \text{Lie}(\mathbb{R}_+^m) : \rho \in \partial(u_0 + tf)(x)\}. \tag{66}$$

We also view $(\rho, \mu_t(\rho))$ as an element of $\text{Lie}(\mathbb{R}_+^m) \oplus \text{Lie}(\mathbf{T}^m)^*$ and define the graph of μ_t over M^o by

$$\mathcal{G}(\mu_t) = \{(\rho, x) \in \text{Lie}(\mathbb{R}_+^m) \times \text{Lie}(\mathbf{T}^m)^* : x \in \partial\psi_t(\rho)\}. \tag{67}$$

As discussed in §2.3, we can cover M with affine charts U_v which straighten out the corner of $M_{\mathbb{R}}$ at each vertex so that it becomes the standard orthant. Hence, we will only discuss the fixed point corresponding to the vertex $v = 0$ of P and the chart $U_0 \simeq \mathbb{C}^m$ in which the Kähler potential has the form (18). Recall also that u_0 is attached to the chart U_0 , and that there are analogous potentials for the other charts U_v .

We now define the ‘exponentiated’ subdifferential by

$$\mathcal{E}(\partial(u_0 + tf)) := \{(x, e^{\rho/2}) \in P \times \text{Lie}(\mathbb{R}_+^m) : \rho \in \partial(u_0 + tf)(x)\}. \tag{68}$$

The basic point is that although $|du_0(x)| = \infty$ for $x \in \partial P$, $\exp \frac{1}{2}du_0(x)$ is well-defined as an element of \mathbb{R}_+^m for x in the corner facets incident at $v = 0$, and indeed $\exp \frac{1}{2}du_0(x)$ has a continuous (in fact, smooth) extension to the corner facets incident at $v = 0$. This is easily checked by writing $u_0 = u_P + g$ where u_P is the canonical symplectic potential and $g \in C^\infty(P)$. As an example, we note that in the case of \mathbb{CP}^1 , $u' = \log \frac{x}{1-x} + g'$ blows up at

$x = 0$ while its exponential $\exp \frac{1}{2}u' = \frac{x}{1-x}e^{g'}$ is well defined and takes the value 0 when $x = 0$ for any g' . Essentially the same calculation verifies that the exponentiated subdifferential has a smooth extension to the boundary for the canonical symplectic potential plus any piecewise linear convex function on the polytope of any toric variety.

PROPOSITION 7.9. $\mathcal{E}(\partial(u_0 + tf)) \subset P \times \mathbb{R}_+^m$ is a C^0 submanifold of T^*P which is homeomorphic to $\mathcal{G}(tf)$. Its boundary consists of the union over open faces F of

$$\mathcal{E}(\partial(u_F + tf)) := \{(x'', e^{\rho''/2}) \in F \times \text{Lie}(\mathbb{R}_+^{m-k}) : \rho'' \in \partial(u_F + tf)(x)\} \quad (69)$$

Proof. To prove this, we split the coordinates into $(x', x'') \in \mathbb{R}^k \times \mathbb{R}^{m-k}$ so that $x' = 0$ defines a given face F , and split the sub-differential vectors in $\partial(u + tf)$ into their x', x'' components to obtain component subdifferentials ∂', ∂'' . To obtain the limit along F^o , we let $x' \rightarrow 0$ while keeping x'' bounded away from zero. Then as $x' \rightarrow 0$ and for $\rho' \in \partial'(u + tf)$, $e^{\frac{1}{2}\rho'} \rightarrow 0 \in \mathbb{R}^k$. Indeed, $e^{\frac{1}{2}\partial' u(x)} \rightarrow 0$ and the addition of $t\partial' f$ only adds a bounded amount to the exponent.

On the other hand, the ‘slices’ $u_0(x', x'') + tf(x', x'')$ for fixed x' , viewed as functions of x'' , have a subdifferential $\partial''(u_0(x', x'') + tf(x', x''))$ in x'' . As is easily seen from the canonical (or Fubini-Study) symplectic potential, the subdifferential is bounded as $x' \rightarrow 0$ as long as x'' stays in the interior of F . Further, $\partial''(u_0(x', x'') + tf(x', x''))$ is a continuously varying C^0 submanifold (or manifold with corners) as x' varies. The same is true if we exponentiate the subdifferential as in (69). Hence as $x' \rightarrow 0$ the exponentiation of $\partial''(u_0(x', x'') + tf(x', x''))$ tends C^0 to (69). Combining with the fact that the exponentiated ∂' subdifferential vanishes as $x' \rightarrow 0$ we conclude that $\mathcal{E}(\partial(u + tf))$ in the interior extends continuously to the corner, where it coincides with (69). \square

We now define a lifted moment map in the chart U_v ; as above, we may assume $v = 0$.

Definition: The exponentiated lifted moment map in U_0 is the map which to $z = e^{\rho/2} \in M^o$ assigns

$$\tilde{\mu}_t(z) = (\mu_t(z), e^{\rho/2}) : U_0 \rightarrow \mathcal{E}(\partial(u + tf)). \quad (70)$$

For $z \in \mu^{-1}(F)$ of the form $z = e^{\rho''/2}$ it assigns

$$\tilde{\mu}_t(z) = (\mu_t(z), e^{\rho''/2}) : U_0 \rightarrow \mathcal{E}(\partial(u + tf)). \quad (71)$$

COROLLARY 7.10. The ‘exponentiated lifted moment map’ $\tilde{\mu}_t(\rho) : U_{0\mathbb{R}} \rightarrow \mathcal{E}(\partial(u + tf))|_{P_o}$ is a homeomorphism. Hence the lifted moment map $\tilde{\mu}_t$ is a homeomorphism $M \rightarrow \mathcal{G}(tf)$.

Proof. By construction, the graph of μ_t on U_v is inverse to the exponentiated subdifferential on its image. \square

Thus we have proved that μ_t is continuous and we have determined its homeomorphism property. The above results also imply the regularity statement of Theorem 7.1

LEMMA 7.11. μ_t is Lipschitz continuous on all of M .

Proof. This follows from Corollaries 7.8 and 7.10: μ_t must be Lipschitz because its graph is the ι -image of the graph of $\partial(u + tf)$ on the interior (and its exponentiation near the boundary). These graphs are manifestly given by piecewise smooth manifolds with corners, hence so is the graph of μ_t . \square

We note that the Lipschitz property of μ_t also follows from a standard result of convex analysis on the open orbit and along the divisor faces M_F : Since $u_0 + tf$ is a convex continuous function on P , its Legendre transform, $\mathcal{L}(u_0 + tf)(\rho)$ is a convex lower semi-continuous function on $\mathbb{R}^m = \text{Lie}(\mathbf{T}^m)$. The same description is true along the boundary faces of P . If $z \in \mu^{-1}(F)$, $I^z = \infty$ unless $x \in F$, so we may restrict the supremum in the Legendre transform to F and it becomes the Legendre transform \mathcal{L}_F along the vector space spanned by F . It follows that $\mathcal{L}_F(u_0 + tf)$ is a convex lower semi-continuous function on $\text{Lie}(\mathbf{T}^m/\mathbf{T}_z^m)$, where \mathbf{T}_z^m is the stabilizer of z .

LEMMA 7.12. *The ρ -gradient map $\nabla\psi_t(\rho)$ is Lipschitz continuous from $\mathbb{R}^m \rightarrow P$. Similarly, $\nabla\psi_t|_{M_F}(\rho'')$ is Lipschitz.*

Proof. The proof is an application of the following fact ([HUL], Theorem 4.2.1):

If $g : \mathbb{R}^m \rightarrow \mathbb{R}$ is strongly convex with modulus $c > 0$, i.e.

$$g(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha g(x_1) + (1 - \alpha)g(x_2) - \frac{c}{2}\alpha(1 - \alpha)\|x_1 - x_2\|^2$$

then ∇g^ is Lipschitz with constant $\frac{1}{c}$, i.e.*

$$\|\nabla g^*(s_1) - \nabla g^*(s_2)\| \leq \frac{1}{c}\|s_1 - s_2\|.$$

On the open orbit, $\psi_t = (u + tf)^*$. We claim that $u + tf$ is strongly convex. Indeed, as in [A, SoZ1], the Hessian $G = \nabla_x^2 u_0$ of the symplectic potential has simple poles on ∂P and is uniformly bounded below, $G \geq cI$ for some $c > 0$ on P . Then,

$$u(\alpha x_1 + (1 - \alpha)x_2) \leq \alpha u(x_1) + (1 - \alpha)u(x_2) - \frac{c}{2}\alpha(1 - \alpha)\|x_1 - x_2\|^2.$$

Since tf is convex, it follows that $u + tf$ is strongly convex with modulus c . It follows that $d\psi_t$ is Lipschitz on M^o and hence that μ_t is.

Using the boundary symplectic potentials, the same proof shows that $\mu_t|_{M_F} : \mu_0^{-1}(F) \rightarrow F$ is Lipschitz continuous for any face F . □

7.6. Explicit formula for μ_t and ψ_t . It is useful to give explicit formulae the moment map μ_t and for ψ_t . First, we give the formula on the inverse image of the smooth domains P_j and then we give the formula on the inverse image of the corner set.

PROPOSITION 7.13. *For each $t \geq 0$, the moment map μ_t defines a diffeomorphism $\mu_{t,j} : e^{t\nu_j}\mu_0^{-1}(P_j) \rightarrow P_j$ given by*

$$\mu_{t,j}(z) = \mu_0(e^{-t\nu_j}z).$$

Further, the union $\bigcup_{j=1}^R e^{t\nu_j}\mu_0^{-1}(P_j)$ is disjoint and therefore

$$\mu_t : \bigcup_{j=1}^R e^{t\nu_j}\mu_0^{-1}(P_j) \rightarrow P \setminus \mathcal{C}$$

is a diffeomorphism with inverse $\mu_t^{-1}(x) = e^{t\nu_j}\mu_0^{-1}(x)$.

Proof. For $x \in P_j$, we have

$$\log \mu_t^{-1}(x) = \nabla u_0(x) + t \nabla f(x) = \nabla u_0 + t \nu_j. \quad (72)$$

Hence,

$$\begin{aligned} \log \mu_t^{-1}(x) &:= \nabla u_t(x) := \nabla u_0 + t \nu_j \\ &= \log \mu_0^{-1}(x) + t \nu_j \\ &= \log(e^{t \nu_j} \mu_0^{-1}(x)) \\ \iff \mu_t^{-1}(x) &= e^{t \nu_j} \mu_0^{-1}(x). \end{aligned}$$

It follows that $x = \mu_t(e^{t \nu_j} \mu_0^{-1}(x))$ and therefore $\mu_t(z) = \mu_0(e^{-t \nu_j} z)$ when $z \in e^{t \nu_j} \mu_0^{-1}(P_j)$.

It is easy to see that images of the smooth regions under $\nabla(u_0 + t f)$ are disjoint (see Proposition 7.6 for details), and therefore $\nabla(u_0 + t f)$ is a diffeomorphism from the smooth chambers to their images, with inverse $\nabla(\psi_t + \varphi_{P^o})$. \square

We now give an analogous formula on the inverse image of the subdifferential lying over the corner set. To give the formula, we introduce some notation. The corner set is a union of hyperplanes of the form

$$H_{jk} = \{x : \lambda_j(x) = \lambda_k(x)\} = \{x : \langle \nu_j - \nu_k, x \rangle = v_k - v_j\}. \quad (73)$$

The inverse image of one hyperplane under μ_0 is the smooth hypersurface of M given by

$$L_{ij} := \mu_0^{-1}(H_{jk}) = \{z \in M : \langle \nu_j - \nu_k, \mu_0(z) \rangle = v_k - v_j\}. \quad (74)$$

PROPOSITION 7.14. *For each $t \geq 0$,*

$$\mu_t^{-1}(\mathcal{C}) = \bigcup_{j,k=1}^R \bigcup_{\xi \in CH(\nu_j, \nu_k)} e^{t\xi} L_{jk}.$$

For $z \in \bigcup_{\xi \in CH(\nu_j, \nu_k)} e^{t\xi} L_{jk}$, we have

$$\mu_t^{jk}(z) = \mu_0(\pi_t^{jk}(z)),$$

where $\mu_t^{jk} : \bigcup_{\xi \in CH(\nu_j, \nu_k)} e^{t\xi} L_{jk} \rightarrow L_{jk}$ is the fibration $e^{t\xi} w \rightarrow w$. Further,

$$\mu_t : \bigcup_{j,k=1}^R \bigcup_{\xi \in CH(\nu_j, \nu_k)} e^{t\xi} L_{jk} \rightarrow \mathcal{G}(\partial(u + t f))|_{\mathcal{C}}$$

is a homeomorphism whose inverse is defined for $(x, \nabla u_0(x) + t \xi) \in \mathcal{G}\partial(u + t f(x))$ by

$$\tilde{\mu}_t^{-1}(x, \xi) = e^{t\xi} \mu_0^{-1}(x).$$

Proof. By Corollaries 7.8 and 7.10, the region $\mu_t^{-1}(H_{jk})$ is parametrized by the map,

$$M_t : (z, \xi) \in L_{jk} \times CH(\nu_j, \nu_k) \rightarrow e^{t\xi} z, \quad (75)$$

where $e^{t\xi} \in \mathbb{R}_+^m$. By (75), it follows that $\mu_t^{-1}(H_{jk})$ fibers over L_{jk} with fibers given by the orbits of $e^{t\xi}$ with $\xi \in CH(\nu_j, \nu_k)$. Then for $z \in \mu_t^{-1}(H_{jk})$, $\pi_t^{jk}(z) = \mu_0^{-1}(\mu_t(z))$.

In the inverse direction, for $x \in \mathcal{C}$ and $\xi \in \partial f(x)$, we have by definition of the lifted moment map $\tilde{\mu}_t$,

$$\iota(2 \log \tilde{\mu}_t^{-1}(x, \xi), x) = (x, \nabla u_0(x) + t\xi), \quad (76)$$

or equivalently,

$$\begin{aligned} 2 \log \tilde{\mu}_t^{-1}(x, \xi) : &= \log \mu_0^{-1}(x) + t\xi \\ &= \log(e^{t\xi} \mu_0^{-1}(x)) \\ \iff \tilde{\mu}_t^{-1}(x, \xi) &= e^{t\xi} \mu_0^{-1}(x). \end{aligned}$$

It follows that $x = \mu_t(e^{t\nu_j} \mu_0^{-1}(x))$ and therefore $\mu_t(z) = \mu_0(e^{-t\nu_j} z)$ when $z \in e^{t\nu_j} \mu_0^{-1}(P_j)$. \square

We observe the analogy between $e^{t\xi} \mu^{-1}(L_{jk})$ and $e^{t\nu_j} \mu^{-1}(P_j)$, and that the union of these domains fills out M . The smooth and corner domains meet along their common boundary,

$$\partial \left(\bigcup_{j=1}^R e^{t\nu_j} \mu_0^{-1}(P_j) \right) = \bigcup_{j,k=1}^R \bigcup_{\xi \in \partial CH(\nu_j, \nu_k)} e^{t\xi} L_{jk}. \quad (77)$$

Now that we have an explicit formula for μ_t we can give simpler formulae for ψ_t than those of (7) and the expressions following. They are stated in Theorem 1.4. For clarity of exposition we restate them in the following

PROPOSITION 7.15. *For any z , $\psi_t(z) = F_t(\mu_t(z)) - I^z(\mu_t(z))$. Hence,*

- (i) *When $z = e^{\rho/2+i\theta} \in \mu^{-1}(P_j)$, then $\psi_t(\rho) = \varphi_{P^o}(\rho - t\nu_j) - t\nu_j - \varphi_{P^o}(\rho)$.*
- (ii) *When $z \in \mu_0^{-1}(F^o)$, then $\psi_t(z) = \varphi_F(\rho'' - t\nu_j'') - t\nu_j - \varphi_F(\rho'')$.*
- (iii) *A point $z = e^{\rho/2+i\theta} \in \mu_0^{-1}(P_j \cap P_k)$ only if $\rho \in tCH(\nu_j, \nu_k)$. In that case, $\mu_t(\rho)$ is a constant point x_0 for $\rho \in tCH(\nu_j, \nu_k)$, and $\psi_t(\rho) = \langle \rho, x_0 \rangle - u(x_0) - t(\langle \nu_j, x_0 \rangle + v_j)$. Analogous formulae hold on the faces of ∂P .*

Proof. In the formula (7) for ψ_t , it is now clear that the supremum is obtained at $x = \mu_t(z)$, giving the first formula.

We can simplify the expression by substituting the expression for I^z . We illustrate with the open orbit: In the open real orbit, $\psi_t(\rho) + \varphi_{P^o}(\rho) = \langle \rho, x_t(\rho) \rangle - (u_0 + tf)(x_t(\rho))$ where $x_t(\rho)$ solves $\rho = \nabla(u_0 + tf)(x_t(\rho))$. On the subdomains P_j , $\nabla(u_0 + tf)$ is a map which inverts μ_t . Hence, $\mu_t(\rho) = x_t(\rho)$. Since $\mu_t(\rho) = \mu_0(\rho - t\nu_j)$ and $f(\mu_0(\rho - t\nu_j)) = \langle \nu_j, \mu_0(\rho - t\nu_j) \rangle - v_j$ in this region,

$$\begin{aligned} \psi_t(\rho) + \varphi_{P^o}(\rho) &= \langle \rho, \mu_t(\rho) \rangle - (u + tf)(\mu_t(\rho)) \\ &= \langle \rho, \mu(\rho - t\nu_j) \rangle - u(\mu(\rho - t\nu_j)) - tf(\mu(\rho - t\nu_j)) \\ &= \langle \rho - t\nu_j, \mu(\rho - t\nu_j) \rangle - u(\mu(\rho - t\nu_j)) - tf(\mu(\rho - t\nu_j)) + t\langle \nu_j, \mu(\rho - t\nu_j) \rangle \\ &= \varphi_{P^o}(\rho - t\nu_j) - t(f(\mu(\rho - t\nu_j)) - \langle \nu_j, \mu(\rho - t\nu_j) \rangle) \\ &= \varphi_{P^o}(\rho - t\nu_j) - tv_j. \end{aligned}$$

Similarly, when $z \in \mu^{-1}(F^o)$, then $\psi_t(z) + \varphi_F = \varphi_F(\rho'' - t\nu_j'')$.

□

Remark: These expressions are consistent with $\psi_t(z) = F_t(\mu_t(z)) - I^z(\mu_t(z))$. For instance, on the open orbit,

$$\psi_t(z) = -tf(\mu_t(z)) - u_0(\mu_t(z)) + \langle \mu_t(z), \log |z| \rangle - \varphi_{P^o}(z), \quad (78)$$

which is the same as (i) by the previous calculation.

As a corollary, we have:

COROLLARY 7.16. *There exist open sets such that $\omega_t^m \equiv 0$.*

Proof. The sets $\mu^{-1}(P_j \cap P_k)$ in (iii) of Proposition 7.15 have non-empty interior. Indeed, they are homeomorphic to the graph of $\partial(u + tf)$ along $P_j \cap P_k$, and hence to the graph of ∂f there. But clearly the graph is a line-segment bundle over a hyperplane and thus has full dimension.

□

7.6.1. Example. We work out the full formula in the case of \mathbb{CP}^1 , with $f(t) = |x - \frac{1}{2}|$. Thus, $\nu_1 = -1, v_1 = \frac{1}{2}; \nu_2 = 1, v_2 = -\frac{1}{2}$. The symplectic potential at time $t > 0$ is $u_t(x) = x \log x + (1-x) \log(1-x) + t|x - \frac{1}{2}|$. Then the subdifferential of u_t is given by

$$\partial u_t(x) = \begin{cases} \log \frac{x}{1-x} - t, & x < \frac{1}{2}, \\ \log \frac{x}{1-x} + t(-1, 1), & x = 0, \\ \log \frac{x}{1-x} + t, & x > \frac{1}{2}. \end{cases}$$

The moment map on the open orbit \mathbb{R} is defined by

$$\mu_t(e^{\rho/2}) = x_t(\rho) : \rho \in \partial u_t(x_t(\rho)),$$

and $\mu_t(\rho) = \frac{1}{2}$ for $\rho \in (-t, t)$. Since $\psi_t(\rho) = \langle \rho, \mu_t(\rho) \rangle - (u + tf)(\mu_t(\rho))$, we have

$$\psi_t(\rho) + \varphi_{P^o}(\rho) = \begin{cases} -\frac{t}{2} + \log(1 + e^{\rho+t}), & \rho \in (-\infty, -t) \\ \frac{\rho}{2} + \log 2 & \rho \in (-t, t) \\ \frac{t}{2} + \log(1 + e^{\rho-t}), & \rho \in (t, \infty) \end{cases}$$

7.6.2. Formula for $\dot{\psi}_t$.

PROPOSITION 7.17. *We have: $\dot{\psi}(t, z) = -f(\mu_t(z))$.*

Proof. We recall that on the open real orbit, $\psi_t(\rho) = \langle \rho, x_t(\rho) \rangle - (u_0 + tf)(x_t(\rho))$ where $x_t(\rho)$ solves $\rho = \nabla(u_0 + tf)(x_t(\rho))$. Hence,

$$\dot{\psi}_t(\rho) = -f(\mu_t(\rho)) + \left(\langle \rho, \frac{d}{dt}x_t(\rho) \rangle - \nabla(u + tf)(x_t(\rho)) \frac{d}{dt}x_t(\rho) \right),$$

and the parenthetical expression vanishes. There are analogous restricted expressions on $\mu^{-1}(F)$ for any boundary facet F , confirming that the identity holds for all $z \in M$.

□

7.7. $\psi_t \in C^{1,1}([0, L] \times M)$. In this section, we complete the proof of the regularity statement in Theorem 1.4. We do this in two steps to separate interior from boundary estimates: first we prove that μ_t is Lipschitz and then we prove that $d\psi_t$ is Lipschitz. The latter improves the former in terms of behavior along \mathcal{D} .

PROPOSITION 7.18. $\mu_t(z)$ is Lipschitz uniformly in $(t, z) \in [0, 1] \times X$.

Proof. In view of Proposition 7.1, we only have to verify that μ_t is uniformly Lipschitz in t . Fix $t = t_0 \geq 0$ and $z_0 \in X$.

(1) Suppose $z_0 \in (\mathbf{C}^*)^n$, $\rho_0 \in \mu_{t_0}^{-1}(P_j^\circ)$ and $x_t = \mu_t(z)$. Then μ_t is smooth in both ρ and t near ρ_0 and t_0 . In fact, $\nabla(u + tf)(x_t) = \rho$ and $D^2(u_0 + tf)(x_t) \cdot \dot{x} = -\nabla f(x_t)$ after taking t -derivative. Therefore

$$\frac{d}{dt}\mu_t(z) = -\left(D^2(u_0 + tf)(x_t)\right)^{-1}\nabla f(x_t) = \left(D^2(u_0)(x_t)\right)^{-1}\nabla f(x_t)$$

and so $\frac{d}{dt}\mu_t(z_0)$ at $t = t_0$ is uniformly bounded in $\mu_{t_0}^{-1}(\cup_j P_j^\circ)$.

(2) Suppose $z_0 \in (\mathbf{C}^*)^n$ and $\rho_0 \in \mathbf{R}^n \setminus \overline{\mu_{t_0}^{-1}(\cup_j P_j^\circ)}$. Then $\rho_0 \in V_{x_{t_0}, t_0}$ and $\mu_t(z_0) = \mu_{t_0}$ for t sufficiently close to t_0 . Hence at $t = t_0$

$$\frac{d}{dt}\mu_t(z_0) = 0.$$

(3) Suppose $z_0 \in (\mathbf{C}^*)^n$ and $\rho_0 \in \partial(\mu_{t_0}^{-1}(\cup_j P_j^\circ))$. One sided derivatives of $\mu_t(z_0)$ exist and fall into the above cases at $t = t_0$.

(4) Suppose $z_0 \in \mathcal{D}$. One only has to restrict μ_t on the subtoric variety and repeat the above argument.

□

We now complete the proof that $\psi_t \in C^{1,1}$ by proving that $d\psi_t$ is Lipschitz. Let us clarify first what this adds to the statement that μ_t is Lipschitz: In terms of the basis $\{\frac{\partial}{\partial \theta_j}\}$ of LT^m , we have (cf. (19)) that

$$\mu_t(z) = (d(\varphi_{P^o} + \psi_t)(\frac{\partial}{\partial \rho_1}), \dots, d(\varphi_{P^o} + \psi_t)(\frac{\partial}{\partial \rho_m})).$$

It follows that $d\psi_t$ is continuous in directions spanned by $\frac{\partial}{\partial \rho_j}, \frac{\partial}{\partial \theta_k}$. However, at $z_0 \in \mathcal{D}$, some of these vector fields vanish, namely those in the infinitesimal isotropy group of z_0 . To show that $d\psi_t$ is a Lipschitz one-form we need to show that $d\psi_t(X)$ is Lipschitz when X is a smooth non-vanishing vector field at $z_0 \in \mathcal{D}$. With no loss of generality, we may straighten out the corner in which $\mu(z_0)$ lies and hence that $\mu(z_0)$ lies in a face of the corner at 0 of the standard orthant. We use the affine coordinates z_j on M adapted to 0 and put $r_j = |z_j|$. Then, $\frac{\partial}{\partial \rho_j} = \frac{1}{2}r_j \frac{\partial}{\partial r_j}$ for $j = 1, \dots, m$. We assume that z_0 lies in the facet of \mathcal{D} defined by $r_1 = \dots = r_k = 0$. Then,

$$d\psi_t(\frac{\partial}{\partial r_j}) = \frac{1}{r_j} \mu_{tj} - d\varphi_{P^o}(\frac{\partial}{\partial r_j}), \quad (79)$$

where μ_{tj} is the j th component of μ_t . Thus, it suffices to check that $\frac{1}{r_\ell} \mu_{t\ell}(r)$ is Lipschitz at $r_1 = \dots = r_k = 0$ for all $\ell = 1, \dots, k$, i.e. that $\frac{1}{r_\ell} d\mu_{t\ell}(r) \in L^\infty$. The Lipschitz property is thus equivalent to

- (1) $\frac{1}{r_\ell^2} \mu_{t\ell}(r) \in L^\infty$, and
- (2) $\frac{1}{r_\ell} d\mu_{t\ell}(r) \in L^\infty$ for each j .

PROPOSITION 7.19. *The 1-form $d\psi_t$ is Lipschitz continuous on M .*

Proof. We have explicit formulae for μ_t away from a codimension subvariety of M . These formulae are sufficient by the following

LEMMA 7.20. *Let X be a compact Kähler manifold. Let Y be a subvariety of X and g a fixed Kähler metric on X . If $\mu \in C^0(X)$ and $|d\mu|_g \leq C$ on $X \setminus Y$ for a uniform constant $C > 0$, then μ is Lipschitz on X .*

Proof. Let z_1, z_2 be two arbitrary points on X . Let $z_{1,\epsilon} \in B_\epsilon(z_1) \cap (X \setminus Y)$ and $z_{2,\epsilon} \in B_\epsilon(z_2) \cap (X \setminus Y)$. Let γ_ϵ be the shortest geodesic joining $z_{1,\epsilon}$ and $z_{2,\epsilon}$ on X . Without loss of generality, we can assume that γ_ϵ only intersects with Y by finite points, say, w_1, \dots, w_k . Let $w_0 = z_{1,\epsilon}$ and $w_{k+1} = z_{2,\epsilon}$. Then

$$|\mu(z_{1,\epsilon}) - \mu(z_{2,\epsilon})| \leq \sum_{j=0}^k |\mu(w_j) - \mu(w_{j+1})| \leq C \sum_j |w_j - w_{j+1}|_g \leq C |\gamma_\epsilon|_g. \quad (80)$$

□

Thus it suffices to have uniform bounds for (1) and (2) on the open sets where we have explicit formulae. We first prove uniform bounds on the smooth domains.

LEMMA 7.21. *With the above notation, $d\mu_{t\ell}$ satisfies the bounds (1)-(2) on the sets $e^{t\nu_j} \mu_0^{-1}(P_j)$ for each j .*

Proof. By Proposition 7.13, on the set $e^{t\nu_j} \mu_0^{-1}(P_j)$ we have $\mu_{t,j}(z) = \mu_0(e^{-t\nu_j} z)$. Properties (1) and (2) above follow immediately because any smooth moment map μ_0 satisfies these estimates, as may be seen from the fact that $\nabla^2 u_0$ has first order poles on the boundary facets.

□

We now prove the bounds in the the complementary set $M \setminus \left(\bigcup_{j=1}^R e^{t\nu_j} \mu_0^{-1}(P_j) \right)$. By Proposition 7.14 it suffices to verify the bounds on each set $e^{t\xi} L_{jk}$ with $\xi \in CH(\nu_j, \nu_k)$.

LEMMA 7.22. *for each j, k $D\mu_t$ satisfies the bounds (1)-(2) in each set $U_{jk} := \{e^{t\xi} L_{jk}, \xi \in CH(\nu_j, \nu_k)\}$.*

Proof. By Proposition 7.14, we have $\mu_t(z) = \mu_0(\pi_t(z))$ where π_t is the fiber map from $U_{jk} \rightarrow L_{jk}$ with fibers the orbits of $e^{t\xi}$. Hence, for each ℓ ,

$$\frac{1}{r_\ell(z)} d\mu_t(z) = \frac{1}{r_\ell(z)} (d\mu_{0\ell})(\pi_t(z)) \circ D\pi_t(z). \quad (81)$$

We now observe that for any compact set K of $\nu \in \mathbb{R}^m$, there exists $C_K \in \mathbb{R}_+$ so that

$$\frac{r_j(e^\nu z)}{r_j(z)} \leq C_K. \quad (82)$$

Indeed, if we write $z = e^{\rho/2}$ then $r_j(z) = e^{\frac{\rho_j}{2}}$ and $\frac{r_j(e^{\nu}z)}{r_j(z)} = e^{\nu_j}$. It follows that

$$\left\| \frac{1}{r_\ell(z)} d\mu_t(z) \right\| \leq C_k \frac{1}{r_\ell(\pi_t(z))} \left\| (d\mu_{0\ell})(\pi_t(z)) \circ D\pi_t(z) \right\|. \quad (83)$$

To complete the proof of the Lemma, we need to show that

- $\frac{1}{r_\ell(w)} d\mu_{0\ell}(w) \in L^\infty(L_{jk})$;
- $D\pi_t(z) \in L^\infty(U_{jk})$.

The first statement holds in a neighborhood of the facet $r_\ell = 0$ of \mathcal{D} and hence holds on its intersection with L_{jk} ; the statement reduces to Lemma 7.11 away from this open set.

Hence the key issue is to prove the boundedness of $D\pi_t$ in U_{jk} . We first note that L_{jk} is the level set $I_{jk} = \nu_k - \nu_j$ of the function

$$I_{jk}(z) := \langle \nu_j - \nu_k, \mu_0(z) \rangle. \quad (84)$$

Furthermore the gradient flow of this function with respect to ω_0 is given by the subgroup $\sigma \rightarrow e^{\sigma(\nu_j - \nu_k)}$ of the \mathbb{R}_+^m action. Indeed, the latter is the joint action of the gradient flows ∇I_j of the action variables I_j which form the components of the moment map $\mu_0 : M \rightarrow P$, i.e. $\mu_0 = (I_1, \dots, I_m)$. This holds because the Hamilton vector field H_{I_j} with respect to ω_0 equals $\frac{\partial}{\partial \theta_j}$ and its image under the complex structure J equals both $\frac{\partial}{\partial \rho_j}$ and ∇I_j where ∇ is the gradient for the metric $g_\omega(X, Y) = \omega(JX, Y)$. Thus, it follows directly from (84) that the gradient flow of I_{jk} is $\sigma \rightarrow e^{\sigma(\nu_j - \nu_k)}$.

Now $\xi \in CH(\nu_j, \nu_k)$ has the form $\xi = \nu_k + s(\nu_j - \nu_k)$ and so $e^{t\xi} = e^{t\nu_k} e^{ts(\nu_j - \nu_k)}$. Thus, the family of hypersurfaces $e^{t\xi} L_{jk}$ for $\xi \in CH(\nu_j, \nu_k)$ is the image under $e^{t\nu_k}$ of the family of hypersurfaces $e^{ts(\nu_j - \nu_k)} L_{jk}$, and the latter is a family of level sets of I_{jk} . In particular, for each t and s , the latter family is orthogonal to the flow lines of $e^{ts(\nu_j - \nu_k)}$ at $e^{ts(\nu_j - \nu_k)} z \in M$ with respect to the Kähler metric $(e^{ts(\nu_j - \nu_k)})^* g_\omega(z) = g_\omega(e^{ts(\nu_j - \nu_k)/2} z)$. This is because $e^{ts(\nu_j - \nu_k)} L_{jk}$ is the level set given by

$$I_{jk}(z; t, s) = \langle \nu_j - \nu_k, \mu_0(e^{ts(\nu_j - \nu_k)/2} z) \rangle.$$

Moreover, $g_\omega(e^{ts(\nu_j - \nu_k)} \cdot)$ is equivalent to $g_\omega(\cdot)$ and it can be seen in \mathbf{R}^n . Suppose

$$I_{jk}(\rho; t, s) = \langle \nu_j - \nu_k, \mu_0(\rho + ts(\nu_j - \nu_k)) \rangle.$$

Hence

$$\nabla_\rho I_{jk}(\rho; t, s) = \nabla_\rho^2 \varphi_0(\rho + ts(\nu_j - \nu_k)) \cdot (\nu_j - \nu_k)$$

and so under the Riemannian metric

$$G_{ts}(\rho) = G_0(\rho + ts(\nu_j - \nu_k)) = \sum_{p,q} \frac{\partial^2 \varphi_0}{\partial \rho_p \partial \rho_q} \varphi_0(\rho + ts(\nu_j - \nu_k)) d\rho_p \otimes d\rho_q$$

on \mathbf{R}^n , and $\nu_j - \nu_k$ is orthogonal to the hypersurface. It is obvious that G_{ts} and G_0 are uniformly equivalent.

It is convenient to slightly modify our problem by removing $e^{t\nu_k}$. In the notation of (75), we change M_t to $\tilde{M}_t : L_{jk} \times [0, 1] \rightarrow M$, where $\tilde{M}_t(z, s) := e^{st(\nu_j - \nu_k)/2} z$. Thus, $M_t(z, \nu_k + s(\nu_j - \nu_k)) = e^{\nu_k t} \tilde{M}_t(z, s)$. We then define $\tilde{\pi}_t : e^{-t\nu_k} U_{jk} \rightarrow L_{jk}$ by $\tilde{\pi}_t \tilde{M}_t(z, s) = z$. Since $\pi_t(w) = \tilde{\pi}_t(e^{-\nu_k t/2} w)$, $D\pi_t$ is bounded on U_{jk} if and only if $D\tilde{\pi}_t$ is bounded on $e^{-t\nu_k} U_{jk}$.

To prove that $D\tilde{\pi}_t$ is bounded on $e^{-t\nu_k}U_{jk}$, we observe that the gradient flow of I_{jk} at fixed s, t takes L_{jk} to another level set of I_{jk} and hence one has orthogonal foliations of $e^{-t\nu_k}U_{jk}$ given by level sets and gradient lines of I_{jk} . We note that the critical points of I_{jk} occur only on \mathcal{D} and by Lemma 7.20 it suffices to bound $D\tilde{\pi}_t$ on its complement. We may thus split the tangent space at each point of $e^{-t\nu_k}U_{jk}$ into $\mathbb{R}\nabla I_{jk} \oplus T\{I_{jk} = C\}$. Since $D\tilde{\pi}_t = 0$ on $\mathbb{R}\nabla I_{jk}$ it then suffices to bound $D(\tilde{\pi}_t|_{\{I_{jk} = C\}})$ uniformly in C as C runs over the levels in $e^{-t\nu_k}U_{jk}$. But $\tilde{\pi}_t|_{\{I_{jk} = C\}}$ is simply the inverse of the map $z \in L_{jk} \rightarrow e^{ts(C)(\nu_j - \nu_k)}z$ where $ts(C)$ is the parameter time of the gradient flow from $I_{jk} = \nu_k - \nu_j$ to the level set $I_{jk} = C$. Hence $D\tilde{\pi}_t$ is bounded above as long as the derivatives of the family of maps $z \rightarrow e^{ts(C)(\nu_j - \nu_k)/2}z$ on L_{jk} have a uniform lower bound. But this is clear since this family forms a compact subset of the group \mathbb{R}_+^m .

This concludes the proof of the Lemma. □

Lemmas 7.21 and 7.22 imply that $d\psi_t$ is Lipschitz, concluding the proof of the Proposition. □

Finally, we consider t derivatives:

PROPOSITION 7.23. *$\dot{\psi}_t$ is Lipschitz on $([0, 1] \times X)$.*

Proof. By Proposition 7.17, and the fact that both f and μ_t are continuous, it follows that $\psi_t \in C^1([0, T] \times X)$. Obviously, $\dot{\psi}_t$ is smooth outside a subvariety of $[0, T] \times X$ and so it suffices to check the uniform Lipschitz condition for $f(\mu_t(\rho))$ in both z and t variables. In the z variables it follows from Proposition 7.18 and the fact that f is Lipschitz.

In the t variable, we note that $\frac{\partial}{\partial t}f(\mu_t(\rho)) = \nu_i \cdot \frac{\partial}{\partial t}\mu_t(\rho)$, and this is bounded by Proposition 7.18. □

Remark:

We note that the geodesic equation

$$\partial_t^2 \varphi_t = |\partial_z \dot{\varphi}_t|_{\omega_t}^2$$

is valid in a weak sense, although both sides are discontinuous, hence ψ_t is a weak solution of the geodesic equation, or equivalently of the Monge-Ampère equation $(\partial\bar{\partial}\Phi)^{m+1} = 0$ where $\Phi(t + i\tau, z) = \psi_t(z)$ (cf. [S, D1] for the relation of the geodesic equation and the Monge-Ampère equation). Since the Monge-Ampère measure is \mathbf{T}^m -invariant, it is equivalent that the real Monge-Ampère measure of ψ_t on $\mathbb{R} \times M_{\mathbb{R}}$ equals zero. In the real domain, a weak solution of the Monge-Ampère equation is a function ψ whose Monge-Ampère measure $M(\psi)$ equals zero, where the Monge-Ampère measure is defined by $M(\psi)(E) = |\partial\psi(E)|$, i.e. by the Lebesgue measure of the image of a Borel set E under the subdifferential map of ψ (see e.g. [CY]).

To see that our ψ_t solves the homogeneous real Monge-Ampère equation, we note that the image of the gradient map of ψ_t is the same as the image of the subdifferential map (in both the t and x variables) of $u + tf$. Since the latter is linear in t , its Monge-Ampère measure in $\mathbb{R} \times P$ equals zero. We conclude that $(\partial_t^2 \varphi_t - |\partial_z \dot{\varphi}_t|_{\omega_t}^2)dt dx$ is the zero measure. It follows that, as measures, $\partial_t^2 \varphi_t dt dx = |\partial_z \dot{\varphi}_t|_{\omega_t}^2 dt dx$.

It follows that $\partial_t^2 \psi_t \in L^\infty$ if and only if $|\partial_z \dot{\psi}_t|_{\omega_t}^2 \in L^\infty$. The metric norm uses the inverse of ω_t , which as observed above vanishes on the open sets $\mu(P_i \cap P_j)$. On the other hand, the formula in Theorem 1.4 shows that $\partial_z \dot{\psi}_t \equiv 0$ there as well.

8. C^1 CONVERGENCE: PROOF OF PROPOSITION 1.5

In this section we prove that $\psi_t(t, z) \rightarrow \psi_t$ in C^1 . The proof uses the properties of the moment map μ_t established in previous sections, and is based on a strong version of Varadhan's Lemma and on uniform large deviations. It follows that $\psi_t \in C^1([0, L] \times M)$ as will be discussed at the end.

8.1. Uniform tilted large deviations upper bound. Our aim is to show:

PROPOSITION 8.1. *For compact sets,*

$$\frac{1}{k} \log \mu_k^{z,t}(K) \leq - \inf_{x \in K} I^{z,t}(x) + o(1), \quad (85)$$

where $o(1)$ denotes a quantity such that $o(1) \rightarrow 0$ as $k \rightarrow \infty$ uniformly in z, t when t lies in a compact set. Here, $I^{z,t}(x) = [I^z(x) - F_t(x)] - \sup_{x \in X} [F_t(x) - I^z(x)]$.

Proof. Since the logarithmic asymptotics of the denominator $Z_k^{z,t}$ in $\mu_k^{z,t}$ follow by the Varadhan's Lemma just proved, it suffices to show that

$$\frac{1}{k} \log \int_K e^{kF_t(x)} d\mu_k^z(x) \leq - \inf_{x \in K} [F_t(x) - I^z(x)] + o(1), \quad k \rightarrow \infty, \quad (86)$$

where the remainder is uniform in $z \in M$ and $t \in [0, L]$ for any $L > 0$. We prove this with a slight generalization of Lemma 6.2. The proof is essentially the same, but for the reader's convenience we include some details.

LEMMA 8.2. *Let $L_k(z, t, \frac{\alpha}{k}) = \frac{1}{k} \log e^{kF_t(\frac{\alpha}{k})} \frac{|s_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} - (F_t - I^z)(\frac{\alpha}{k})$. Then $L_k(z, t, \frac{\alpha}{k}) = -\frac{1}{k} \log Q_{h^k}(\alpha) + u(\frac{\alpha}{k})$ for $z \in M^o$ and satisfies*

$$L_k(z, t, \frac{\alpha}{k}) = O\left(\frac{1}{k}\right), \quad (k \rightarrow \infty)$$

uniformly in $z \in M^o$, $t \in [0, L]$ and $\alpha \in kP$. The same formula and remainder hold when $\mu_t(z) \in F$ and $\frac{\alpha}{k} \in F$.

Proof. First assume that $z = e^{\rho/2} \in M^o$. Then,

$$\frac{1}{k} \log e^{kF_t(\frac{\alpha}{k})} \frac{|s_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} = F_t(\frac{\alpha}{k}) + \langle \frac{\alpha}{k}, \rho \rangle - \varphi(\rho) - \frac{1}{k} \log Q_{h^k}(\alpha), \quad (87)$$

while

$$(F_t - I^z)(\frac{\alpha}{k}) = F_t(\frac{\alpha}{k}) - \langle \frac{\alpha}{k}, \rho \rangle + u(\frac{\alpha}{k}) + \varphi(\rho). \quad (88)$$

Hence,

$$\frac{1}{k} \log \frac{|s_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} - (F_t - I^z)(\frac{\alpha}{k}) = -\frac{1}{k} \log Q_{h^k}(\alpha) + u(\frac{\alpha}{k}). \quad (89)$$

The rest continues as in the case $t = 0$. \square

The following Lemma, a generalization of Proposition 6.3, concludes the proof:

LEMMA 8.3. *For any compact subset $K \subset \bar{P}$, we have the uniform upper bound*

$$\frac{1}{k} \log \mu_k^{z,t}(K) \leq - \inf_{x \in K} I^{z,t}(x) + O\left(\frac{\log k}{k}\right),$$

where the remainder is uniform in z and $t \in [0, L]$.

Proof. Taking into account the denominator in $\mu_k^{z,t}$ and Lemma 8.2, the proof as in Lemma 6.3 leads to the conclusion,

$$\frac{1}{k} \log \mu_k^{z,t}(K) = \frac{1}{k} \log \sum_{\alpha \in kP: \frac{\alpha}{k} \in K} e^{-kI^{z,t}(\frac{\alpha}{k})} + O\left(\frac{1}{k}\right) + O\left(\frac{\log k}{k}\right)$$

and the rest of the proof goes as before. \square

\square

8.2. Proof of Proposition 1.5.

We first prove:

LEMMA 8.4. $\mu_t(z) = \lim_{k \rightarrow \infty} x d\mu_k^{z,t}$ and $\mu_k^{z,t} \rightarrow \delta_{\mu_t(z)}$.

Proof. We first use [E], Theorem II.6.3 (particularly the proof that $\mathbf{E}(W_n/a_n) \rightarrow \nabla c(0)$ on page 49) to show that $\mu_k^{z,t} \rightarrow \delta_{\mu_t(z)}$ in the weak sense as $k \rightarrow \infty$.

The proof uses the logarithmic moment generating function for $d\mu_k^{z,t}$, defined as before by

$$\begin{aligned} \Lambda^{z,t}(\rho) &= \lim_{k \rightarrow \infty} \frac{1}{k} \log \int_P e^{k\langle x, \rho \rangle} d\mu_k^{z,t} \\ &= \lim_{k \rightarrow \infty} \frac{1}{k} \log \int_P e^{k(\langle x, \rho \rangle + F_t(x))} d\mu_k^z - \frac{1}{k} \log Z_k^{z,t}. \end{aligned}$$

By Varadhan's Lemma, the first term tends to

$$\sup_{x \in P} (\langle x, \rho \rangle + F_t(x) - I^z(x))$$

and the second tends to

$$\sup_{x \in P} (F_t(x) - I^z(x)).$$

Thus, we have

$$\Lambda^{z,t}(\rho) = \sup_{x \in P} (\langle x, \rho \rangle + F_t(x) - I^z(x)) - \sup_{x \in P} (F_t(x) - I^z(x)). \quad (90)$$

Up to the constant Rt , the first term defines the Legendre transform of the strictly convex function $I^z + tf$ and since the second term is constant in ρ , $\Lambda^{z,t}(\rho)$ is a strictly convex function of ρ , which up to a constant equals

$$\begin{cases} \mathcal{L}(u_0 + tf)(\rho + \log |z|) = \psi_t(\rho + \log |z|) + \varphi_{P^o}(\log |z|), & z \in M^o; \\ \mathcal{L}_F(u_0 + tf)(\rho'' + \log |z''|) = \psi_t(\rho'' + \log |z''|) + \varphi_F(|z''|), & z \in M_F. \end{cases} \quad (91)$$

In the evaluation on F we note that the supremum is taken for $x \in F$ and hence $\langle x, \rho \rangle = \langle x'', \rho'' \rangle$ where ρ'' is the component of ρ along F .

By Proposition 7.2, $\Lambda^{z,t}(\rho)$ is a C^1 function of $\rho \in \mathbb{R}^m$ for all $z \in M^o$ and it is a C^1 function of $\rho'' \in \mathbb{R}^{m-k}$ for $z \in M_F$. Then by [E], Theorem II.6.3 (particularly the proof that $\mathbf{E}(W_n/a_n) \rightarrow \nabla c(0)$ on page 49) it follows that

$$\lim_{k \rightarrow \infty} \int_P x d\mu_k^{z,t} \rightarrow \begin{cases} \nabla(\psi_t + \varphi_{P^o})(\log|z|), & z \in M^o; \\ \nabla''(\psi_t + \varphi_{P^o})(\log|z''|), & z \in M_F. \end{cases}$$

By Definition (63), the limit equals $\mu_t(z)$ in all cases, and Theorem II. 6. 3 of [E] shows that $\mu_k^{t,z} \rightarrow \delta_{\mu_t(z)}$. \square

8.3. C^1 Convergence and Varadhan's Lemma. We now use the large deviations principle of the previous section to prove that the limit in Lemma 8.4 is C^0 and at the same time to complete the proof of Proposition 1.5.

Proof. Each first derivative has the form

$$\int_{\bar{P}} \psi(x) d\mu_k^{t,z}(x),$$

where $d\mu_k^{t,z}$ is the time-tilted measure (9) and $\psi \in C(\bar{P})$. Indeed, on the open orbit,

$$\begin{aligned} \frac{1}{k} \nabla_\rho \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} e^{2tk(R-f(\frac{\alpha}{k}))} \frac{|s_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} &= \frac{\sum_{\alpha \in kP \cap \mathbb{Z}^m} \left(\frac{\alpha}{k} - \mu(z)\right) e^{2tk(R-f(\frac{\alpha}{k}))} \frac{|s_\alpha|_{h^k}^2}{Q_{h^k}(\alpha)}}{\sum_{\alpha \in kP \cap \mathbb{Z}^m} e^{2tk(R-f(\frac{\alpha}{k}))} \frac{|s_\alpha|_{h^k}^2}{Q_{h^k}(\alpha)}} \\ &= \int (x - \mu(z)) d\mu_k^{z,t}(x) \end{aligned}$$

Near \mathcal{D} we cannot express derivatives in terms of ∇_ρ but must rather use ∇_z as in [SoZ2]. This only has the effect of changing the sum over α to a sum over $\alpha_n \neq 0$ and then changing $\alpha \rightarrow \alpha - (0, \dots, 1_n, \dots, 0)$.

Also, the ∂_t derivative has the form

$$\begin{aligned} \frac{1}{k} \frac{\partial}{\partial t} \log \sum_{\alpha \in kP \cap \mathbb{Z}^m} e^{2tk(R-f(\frac{\alpha}{k}))} \frac{|S_\alpha(z)|_{h^k}^2}{Q_{h^k}(\alpha)} &= \frac{\frac{1}{k} \sum_\alpha e^{2tk(R-f(\frac{\alpha}{k}))} (2k(R-f(\frac{\alpha}{k})) \frac{e^{k\langle \alpha, \rho \rangle}}{Q_{h^k}(\alpha)})}{\left(\sum_\alpha e^{2tk(R-f(\frac{\alpha}{k}))} \frac{e^{k\langle \alpha, \rho \rangle}}{Q_{h^k}(\alpha)} \right)} \\ &= \int_P (2(R - f(x))) d\mu_k^{z,t}(x). \end{aligned} \quad (92)$$

Hence to prove the result we need only to show:

LEMMA 8.5. *For any continuous $\psi \in C(\bar{P})$, $\int_{\bar{P}} \psi(x) d\mu_k^{t,z}(x) = \psi(\mu_t(z)) + o(1)$, with uniform remainder.*

To prove the Lemma we first note that $I^{z,t}(x)$ attains its infimum at a unique point $\bar{x}(z, t) = \mu_t(z)$ where μ_t is the moment map of ψ_t .

We have,

$$\int_{\bar{P}} \psi(x) d\mu_k^{t,z}(x) = \int_{|x-\mu_t(z)| \leq \epsilon} \psi(x) d\mu_k^{t,z}(x) + \int_{|x-\mu_t(z)| > \epsilon} \psi(x) d\mu_k^{t,z}(x).$$

By Proposition 8.1, the second term is bounded by

$$\begin{aligned} \max_P |\psi| \cdot \mu_k^{z,t}(|x - \mu_t(z)| > \epsilon) &\leq \max_P |\psi| \cdot e^{-k(\inf_{|x-\mu_t(z)|>\epsilon} I^{z,t}(x) + o(1))} \\ &= o(1), \end{aligned}$$

where the remainder $o(1)$ is uniform for any $\epsilon > 0$. Since $\inf_{z \in M, t \in [0, L]} \inf_{|x-\mu_t(z)|>\epsilon} I^{z,t}(x) > 0$, the final estimate is uniform in t, z .

We then consider the first term. Since ψ is uniformly continuous on \bar{P} , there exists ϵ for any given $\delta > 0$ so that $|\psi(x) - \psi(\mu_t(z))| \leq \delta$ if $|\mu_t(z) - x| \leq \epsilon$. The first term is then $\psi(\mu_t(z))$ plus $O(\delta)$. Choosing δ sufficiently small and then k sufficiently large completes the proof of the Lemma. \square

Note that this gives another proof that $\psi_t \in C^1([0, L] \times M)$.

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